

Derived bi-duality via homotopy limit

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Abstract

We show that a derived bi-duality dg-module is quasi-isomorphic to the homotopy limit of a certain tautological functor. This is a simple observation, which seems to be true in wider context. From the view point of derived Gabriel topology, this is a derived version of results of J. Lambek about localization and completion of ordinary rings. However the important point is that we can obtain a simple formula for the bi-duality modules only when we come to the derived world from the abelian world.

We give applications. 1. we give a generalization and an intuitive proof of Efimov-Dwyer-Greenlees-Iyenger Theorem which asserts that the completion of commutative ring satisfying some conditions is obtained as a derived bi-commutator. (We can also prove Koszul duality for dg-algebras with Adams grading satisfying mild conditions.) 2. We prove that every smashing localization of dg-category is obtained as a derived bi-commutator of some pure injective module. This is a derived version of the classical results in localization theory of ordinary rings.

These applications shows that our formula together with the viewpoint that a derived bi-commutator is a completion in some sense, provide us a fundamental understanding of a derived bi-duality module.

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1 Introduction

The following situation and its variants are ubiquitous in Algebras and Representation theory:

Let R be a ring, J an R -module and $E := \text{End}_R(J)^{\text{op}}$ the opposite ring of the endomorphism ring of J over R . Then we have the duality

$$(-)^* := \text{Hom}_R(-, J) : \text{Mod } R \rightleftarrows (\text{Mod } E)^{\text{op}} : \text{Hom}_E(-, J) =: (-)^*$$

and the unite map $\epsilon_M : M \rightarrow M^{**}$ is given by the evaluation map:

$$\epsilon_M(m) : \text{Hom}_R(M, J) \rightarrow J, f \mapsto f(m) \text{ for } m \in M.$$

The bi-dual R^{**} of R is called the *bi-commutator* (or the *double centralizer*) and denoted by $\text{Bic}_R(J)$. The following is more popular expression (or the usual definition) of the bi-commutator

$$\text{Bic}_R(J) := \text{End}_E(J)^{\text{op}}.$$

The bi-commutator has a ring structure and the evaluation map $\epsilon_R : R \rightarrow \text{Bic}_R(J)$ become a ring homomorphism. In particular, the case when the canonical algebra homomorphism $R \rightarrow \text{Bic}_R(J)$ become an isomorphism, the module J is said to have the *double centralizer property*. Dualities together with evaluation maps, bi-commutators and double centralizer properties are one of the central topics in Algebras and Representation theory. (See e.g. [10, 16, 17, 18, 24, 35])

Recently the concern with the *derived bi-commutators* (or the *derived double centralizers*) has been growing:

Let R a ring (or more generally dg-algebra) J an (dg-) R -module and $\mathcal{E} := \mathbb{R}\text{End}_R(J)^{\text{op}}$ the opposite dg-algebra of the endomorphism dg-algebra of J . Then the derived bi-commutator is defined by

$$\mathbb{B}\text{ic}_R(J) := \mathbb{R}\text{End}_{\mathcal{E}}(J)^{\text{op}}.$$

There also exists a canonical algebra homomorphism $R \rightarrow \mathbb{B}\text{ic}_R(J)$. Derived double centralizer property for special modules has been extensively studied as a part of Koszul duality. (See e.g. [19, 29].)

In [3, Section 4.16], Dwyer-Greenlees-Iyenger call a pair (R, J) *dc-complete*, in the case when J has derived double centralizer property. They proved the following surprising and impressive theorem, which we will refer as *completion theorem*.

Theorem 1.1 ([3],[5]). *Let R be a commutative Noetherian ring and \mathfrak{a} an ideal such that the residue ring R/\mathfrak{a} is of finite global dimension. We denote by \widehat{R} the \mathfrak{a} -adic completion. Then we have a quasi-isomorphism*

$$\widehat{R} \simeq \mathbb{B}\text{ic}_R(R/\mathfrak{a})$$

where $\mathbb{B}\text{ic}_R(R/\mathfrak{a})$ is the derived bi-commutator of R/\mathfrak{a} over R .

From the view point of Derived-Categorical Algebraic Geometry (DCAG), all important procedure in Algebraic Geometry should have derived-categorical interpretation. In [14] Kontsevich proposed that formal completion for a scheme is obtained as a derived bi-commutator. Following this idea, Efimov [5] introduced derived bi-commutator of subcategory $\mathcal{J} \subset \mathcal{D}(R)$ and proved a scheme version of above theorem. Since formal completion plays an important role in Algebraic Geometry, completion theorem and its scheme version are expected to become important in DCAG. Therefore it is desirable to obtain better understanding of this theorem.

In the proof of completion theorem, Grothendieck vanishing theorem for local cohomology is used. Since it is special theorem for commutative Noetherian rings, it is preferable to obtain more categorical proof. Recently Porta-Shaul-Yekutieli [28] generalize completion theorem for a commutative ring R and a weakly proregular ideal \mathfrak{a} based on their work [27] about the derived functors of the completion functors and the torsion functors. However it is still remain unclear that to what extent we can obtain a transcendental outcome by a homological operation with finite input. In this paper we establish a simple description of the derived bi-commutator, which enable us to give a more intuitive proof of completion theorem. Actually the description is given by a certain tautological homotopy limit, and hence seems to state that every derived bi-commutator is completion in some sense. (We can make this precise by introducing a notion of derived Gabriel topology.)

For this purpose, we study derived bi-duality:

$$(-)^{\otimes} := \mathbb{R} \operatorname{Hom}_R(-, J) : \mathcal{D}(R) \rightleftarrows \mathcal{D}(\mathcal{E})^{\operatorname{op}} : \mathbb{R} \operatorname{Hom}_{\mathcal{E}}(-, J) =: (-)^{\otimes}.$$

For a special class of modules J , derived bi-duality is already studied in the context of Gorenstein dg-algebras [6, 11, 21]. We consider general dg-module J and establish a simple description of the derived bi-dual module $M^{\otimes\otimes}$ via a certain tautological homotopy limit. This is the main theorem of this paper. As an application other than completion theorem, we discuss smashing localization of dg-categories.

As mentioned above, derived bi-dualities, derived bi-commutator and derived double centralizer property are expected to play prominent roles in Algebras, Representation theory, Derived-Categorical Algebraic Geometry. Our main theorem together with the view point that derived bi-commutators are completion in some sense, would have many applications. Moreover since the main theorem is proved in a formal argument, the same formula should hold in more wider context. Bi-duality is a basic operation which appears in every where of mathematics. So it can be expect that our main theorem become an indispensable tool in many area of mathematics.

Below we explain the contents of the present paper. First, we give the main theorem with omitting some details. Next, we show that the main theorem leads to an intuitive proof of a generalization of Theorem 1.1.

1.1 Derived bi-duality via homotopy limit

Let \mathcal{A} be a dg-algebra and J a dg \mathcal{A} -module. We denote $\mathcal{E} := (\mathbb{R} \operatorname{End}_{\mathcal{A}}(J))^{\operatorname{op}}$ be the opposite dg-algebra of the endomorphism dg-algebra. Then J has a natural dg \mathcal{E} -module structure. We obtain the dualities

$$(-)^{\otimes} := \mathbb{R} \operatorname{Hom}_{\mathcal{A}}(-, J) : \mathcal{D}(\mathcal{A}) \rightleftarrows \mathcal{D}(\mathcal{E})^{\operatorname{op}} : \mathbb{R} \operatorname{Hom}_{\mathcal{E}}(-, J) =: (-)^{\otimes}$$

There are natural transformations $\epsilon : 1_{\mathcal{D}(\mathcal{A})} \rightarrow (-)^{\otimes\otimes}$ induced from evaluation morphisms.

We denote by $\langle J \rangle$ the smallest thick subcategory containing J . Namely $\langle J \rangle$ is the full subcategory of $\mathcal{D}(\mathcal{A})$ consisting those objects which constructed from J by taking cones, shifts, and direct summands finitely many times.

Let M be a dg \mathcal{A} -module. We denote by $\langle J \rangle_{M/}$ the under category. Namely, the objects of $\langle J \rangle_{M/}$ are morphisms $k : M \rightarrow K$ with $K \in \langle J \rangle$ and the morphisms from $k : M \rightarrow K$ to $\ell : M \rightarrow L$ are the

morphisms $\phi : K \rightarrow L$ in $\langle J \rangle$ such that $\ell = \phi \circ k$. This category $\langle J \rangle_{M/}$ comes naturally equipped with the co-domain functor $\Gamma : \langle J \rangle_{M/} \rightarrow \mathcal{D}(\mathcal{A})$ which sends an object $k : M \rightarrow K$ to its co-domain K .

$$\Gamma : \langle J \rangle_{M/} \rightarrow \mathcal{D}(\mathcal{A}), \quad [k : M \rightarrow K] \mapsto K.$$

An elementary observation given in Appendix A.1 suggest the following simple formula for the derived bi-dual module $M^{\otimes\otimes}$.

“Theorem” 1.2. *We have the following quasi-isomorphism*

$$M^{\otimes\otimes} \simeq \operatorname{holim}_{\langle J \rangle_{M/}} \Gamma$$

Remark 1.3. In the above theorem and the following corollary we omit homotopy theoretical details. For the rigorous statement see Theorem 3.4.

Since the bi-dual $\mathcal{A}^{\otimes\otimes}$ of \mathcal{A} is naturally isomorphic to the derived bi-commutator $\mathbb{B}ic_{\mathcal{A}}(J)$,

$$\mathcal{A}^{\otimes\otimes} = \mathbb{R} \operatorname{Hom}_{\mathcal{E}}(\mathbb{R} \operatorname{Hom}_{\mathcal{A}}(\mathcal{A}, J), J) \cong \mathbb{R} \operatorname{Hom}_{\mathcal{E}}(J, J) = \mathbb{B}ic_{\mathcal{A}}(J)$$

in particular, we have the following corollary.

“Corollary” 1.4.

$$\mathbb{B}ic_{\mathcal{A}}(J) \simeq \operatorname{holim}_{\langle J \rangle_{\mathcal{A}/}} \Gamma.$$

These theorem and corollary provide us a fundamental understanding of derived bi-duality functors. We give applications in this paper.

1.2 Completion via derived bi-commutator

As the first application, we generalize the completion theorem and give an intuitive proof.

Let R be a ring and \mathfrak{a} a two-sided ideal. An (right) R -module M is called \mathfrak{a} -torsion if for any $m \in M$ there exists $n \in \mathbb{N}$ such that $m\mathfrak{a}^n = 0$. We denote by $\mathfrak{a}\text{-tor}$ the full subcategory of $\operatorname{Mod} R$ consisting of \mathfrak{a} -torsion modules. We denote by $\mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$ the full subcategory of $\mathcal{D}(R)$ consisting of complexes with \mathfrak{a} -torsion cohomology groups. We denote by $\mathcal{D}(\mathfrak{a}\text{-tor})$ the full subcategory of $\mathcal{D}(R)$ consisting of complexes each term of which is \mathfrak{a} -torsion module.

Theorem 1.5. *Assume that the canonical inclusion functor $\mathcal{D}(\mathfrak{a}\text{-tor}) \rightarrow \mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$ gives an equivalence and that R/\mathfrak{a}^n belongs to $\langle R/\mathfrak{a} \rangle$ for $n \geq 0$. We denote by \widehat{R} the \mathfrak{a} -adic completion. Then we have a quasi-isomorphism*

$$\mathbb{B}ic_R(R/\mathfrak{a}) \simeq \widehat{R}.$$

Remark 1.6. We put artificial conditions on the above theorem, in order to clarify that to what extent the derived bi-commutator gives the \mathfrak{a} -adic completion.

“Proof”.

Assumption 1.7. *In this “Proof” we assume that $\operatorname{holim} = \lim$.*

We denote by \mathcal{I} the (non-full) subcategory of $\langle R/\mathfrak{a} \rangle_{R/}$ which consists of objects $\pi^n : R \rightarrow R/\mathfrak{a}^n$ for $n \geq 1$ and of morphisms $\pi^m \rightarrow \pi^n$ induced from the canonical projections $\varphi^{m,n} : R/\mathfrak{a}^m \rightarrow R/\mathfrak{a}^n$ for $m \geq n$. In other words, \mathcal{I} is the image of the functor $(\mathbb{Z}_{\geq 1})^{\operatorname{op}} \rightarrow \mathcal{D}(\mathcal{A})$ which sends an object n to π^n and a morphism $m \rightarrow n$ to $\pi^m \rightarrow \pi^n$ where we consider the ordered set $\mathbb{Z}_{\geq 1}$ as a category in the standard way. Then we have

$$\lim_{\mathcal{I}} \Gamma|_{\mathcal{I}} \cong \lim_{n \rightarrow \infty} R/\mathfrak{a}^n \cong \widehat{R}.$$

Therefore by Corollary 1.4 and Assumption 1.7, it is enough to show that $\lim_{\mathcal{I}} \Gamma|_{\mathcal{I}} \cong \lim_{\langle R/\mathfrak{a} \rangle_{R/}} \Gamma$. Therefore it is enough to show that \mathcal{I} is a left cofinal subcategory of $\langle R/\mathfrak{a} \rangle_{R/}$. Namely only we have to show that the over category $\mathcal{I}_{/k}$ is non-empty and connected for each $k \in \langle R/\mathfrak{a} \rangle_{R/}$.

Let $k : R \rightarrow K$ be an object of $\langle R/\mathfrak{a} \rangle_{R/}$. Since we assume that $\mathcal{D}(\mathfrak{a}\text{-tor}) \xrightarrow{\sim} \mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$, K belongs to $\mathcal{D}(\mathfrak{a}\text{-tor})$. It follows that K is quasi-isomorphic to a complex K' each terms of which is an \mathfrak{a} -torsion modules. Therefore a morphism $k : R \rightarrow K$ canonically factors through some cyclic \mathfrak{a} -torsion module R/\mathfrak{a}^n .

$$\begin{array}{ccc} R & & \\ \pi^n \downarrow & \searrow k & \\ R/\mathfrak{a}^n & \xrightarrow{\psi} & K \end{array}$$

In other words there exists a morphism $\psi : \pi^n \rightarrow k$ in $\langle R/\mathfrak{a} \rangle_{R/}$. This proves the non-emptiness of $\mathcal{I}_{/k}$. Since the factorization $k = \psi \circ \pi^n$ is canonical, we see that $\mathcal{I}_{/k}$ is connected. This shows that \mathcal{I} is left co-final in $\langle R/\mathfrak{a} \rangle_{R/}$ and completes the “proof”. \square

1.3 Smashing localization via derived bi-commutator

First we recall the following classical fact.

Theorem 1.8 ([17, Corollary 3.4.1], [24, Theorem 7.1]). *Let $f : R \rightarrow S$ be a (right) Gabriel localization of a ring R , that is, f is an epimorphism in the category of rings and S is left flat over R . Let J be a co-generator of the torsion theory which corresponds to the Gabriel localization f . If we take a product $J' := J^\kappa$ of copies of J over large enough cardinal κ , then we have an isomorphism*

$$\text{Bic}_R(J') \cong S.$$

In this section we prove a derived version. A morphisms $f : \mathcal{A} \rightarrow \mathcal{B}$ of dg-algebras is called *smashing localization* if the restriction functor $f_* : \mathcal{D}(\mathcal{B}) \rightarrow \mathcal{D}(\mathcal{A})$ is fully faithful. Recall that a ring homomorphism $R \rightarrow S$ is an epimorphism in the category of rings if and only if the restriction functor $f_* : \text{Mod } S \rightarrow \text{Mod } R$ is fully faithful. Therefore smashing localization can be considered as a dg-version of epimorphisms.

Theorem 1.9. *Let $\mathcal{A} \rightarrow \mathcal{B}$ be a smashing localization of dg-algebras and J be a pure injective co-generator of $\mathcal{D}(\mathcal{B})$. Then we have a quasi-isomorphism over \mathcal{A}*

$$\text{Bic}_{\mathcal{A}}(f_* J') \simeq \mathcal{B}.$$

where $J' = J^{\Pi\kappa}$ is a large enough product of J .

The notion of pure injective co-generator which is introduced by Krause [15] is a dg-version of injective co-generator for the module category. We will recall the definition in Definition 7.2.

A similar theorems is proved by Nicolás and Saorin [25]. In our way of the proof, an essential point is the following theorem, which is also intuitively proved from the view point that a bi-duality is a completion in some sense (See Appendix A.2).

Theorem 1.10. *Let J a pure injective co-generator of $\mathcal{D}(\mathcal{A})$ and M a dg \mathcal{A} -module. If we take a product $J' = J^{\Pi\kappa}$ of copies of J over large enough cardinal κ , then the evaluation morphism is a quasi-isomorphism*

$$\epsilon_M : M \xrightarrow{\sim} M^{\otimes\otimes}$$

where the bi-dual is taken over J' .

In the case when \mathcal{A} is an ordinary ring and M is a module, the same results is already proved by Shamir [32].

1.4 Koszul duality for Adams graded dg-algebras (a part of joint work with A. Takahashi)

The following theorem will be proved and applied in [23].

Theorem 1.11. *Let $\mathcal{A} := \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots$ be an \mathbb{N} -Adams graded dg-algebra. If the \mathcal{A}_0 -modules \mathcal{A}_n satisfies a mild condition. Then we have a quasi-isomorphism*

$$\mathbb{B}ic_{\mathcal{A}}(\mathcal{A}/\mathcal{A}_{\geq 1}) \simeq \mathcal{A}$$

The proof is given in the same way of the proof of Theorem 1.5.

1.5 From the view point of Derived Gabriel topology

Gabriel topology is a special class of linear topology on rings, which plays an important role in the theory of localization of rings [33]. The notion of derived Gabriel topology, which is a Gabriel topology for a dg-algebra, is introduced in [22]. From the view point of derived Gabriel topology, Theorem 1.2 says that the derived bi-dual $M^{\otimes\otimes}$ equipped with “the finite topology” is the “ J -adic completion” of M . In this sense Theorem 1.2 is inspired by the following results of J. Lambek.

Theorem 1.12 ([17, Theorem 4.2], (See also [18, Theorem 3.7])). *Let R be a ring and J an injective R -module. For an R -module M , we denote by $Q(M)$ the module of quotients with respect to J . Assume that every torsionfree factor module of $Q(M)$ is J -divisible. Then the (ordinary) bi-duality $\text{Hom}_{\text{End}_R(J)}(\text{Hom}_R(M, J), J)$ equipped with the finite topology is the J -adic completion of $Q(M)$.*

We remark that in the original Lambek Theorem, there are several assumptions. Contrary to this, the main theorem has no assumptions. This is an important difference from an usual derived version of some result in classical theory of rings and modules. So the point is that we can obtain a simple formula for the bi-duality modules only when we come to the derived world from the abelian world.

At the first sight, three theorems below concerning on derived bi-dualities

- Completion theorem
- Localization theorem
- Koszul duality

seem to be theorems of different kind. However in the present paper we will see that these are consequences of a simple formula, which is the main theorem 1.2. From the view point of derived Gabriel topology, these theorems are consequences of completeness of each algebras with respect to appropriate topologies.

Notations and conventions

Unless otherwise is stated the term “modules” means right modules.

Let M and N be modules. Assume that direct sum decompositions $M = \bigoplus M_j$ and $N = \bigoplus N_i$ are given. We often use a matrix style expression of morphisms $(f_{i,j}) : M \rightarrow N$. In this paper no modules are given more than one direct sum decompositions. Hence there are no danger of confusion. For the sake of space, we often use the transposed expression ${}^t(f_1, \dots, f_n) : M \rightarrow N_1 \oplus \cdots \oplus N_n$ of

a column vector $\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} : M \rightarrow N_1 \oplus \cdots \oplus N_n$.

The totally ordered sets $(\mathbb{Z}_{\geq 1})^{\text{op}}$ and $(\mathbb{Z}_{\geq 0})^{\text{op}}$ are always considered as categories with a unique morphism $\mathbf{geq}^{m,n} : m \rightarrow n$ for $m \geq n$.

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2 Preliminaries

2.1 Simplicial model categories

For the theory of simplicial model categories, we refer to [8, 9, 20].

• **Model categories.** For a model category \mathcal{M} , we denote by \mathcal{M}° the full subcategory of \mathcal{M} consisting of fibrant-cofibrant objects. For a full subcategory $\mathcal{N} \subset \mathcal{M}$ we denote by $\text{Ho}(\mathcal{N})$ the full subcategory of the homotopy category $\text{Ho}(\mathcal{M})$ of \mathcal{M} spanned the objects of \mathcal{N} .

Recall the model structure in \mathcal{M} induces the model structure in the under category $\mathcal{M}_{M/}$ in the following way: let $\varphi : k \rightarrow \ell$ a morphism of $\mathcal{M}_{M/}$ from $k : M \rightarrow K$ to $\ell : M \rightarrow L$. We denote by $\varphi' : K \rightarrow L$ the morphism of \mathcal{M} induced from φ . Then φ is defined to be weak equivalence (resp. fibration, cofibration) in $\mathcal{M}_{M/}$ if φ' is so in \mathcal{M} . In particular we see that an object $k : M \rightarrow K$ is fibrant if and only if the morphism k is a fibration in \mathcal{M} and that k is cofibrant if and only if the domain P is cofibrant in \mathcal{M} .

In a similar way the model structure in \mathcal{M} induces the model structure in the over category $\mathcal{M}_{M/}$. There is a natural functor $\text{Ho}(\mathcal{M}_{M/}) \rightarrow \text{Ho}(\mathcal{M})_{M/}$. In general we can't expect that this functor is full. However we have the following lemma.

Lemma 2.1. *Let \mathcal{M} be a model category, M a cofibrant object, K a fibrant-cofibrant object and L a fibrant object. Assume that a cofibration $k : M \rightarrow K$ and a morphism $\ell : M \rightarrow L$ are given. Assume moreover that a morphism $\overline{\varphi} : k \rightarrow \ell$ in $\text{Ho}(\mathcal{M})_{M/}$ is given. In other words the following commutative diagram is given in the homotopy category $\text{Ho}(\mathcal{M})$:*

$$\begin{array}{ccc} M & & \\ k \downarrow & \searrow \ell & \\ K & \xrightarrow{\overline{\varphi}} & L. \end{array}$$

Then there is a morphism $\varphi : k \rightarrow \ell$ in $\mathcal{M}_{M/}$ such that the image of φ in $\text{Ho}(\mathcal{M})$ coincide with $\overline{\varphi}$. Namely there exists a morphism $\varphi : K \rightarrow L$ such that $\ell = \varphi \circ k$ and the image of φ in $\text{Ho}(\mathcal{M})$ coincide with $\overline{\varphi}$.

Proof. There exists a morphism $\varphi' : K \rightarrow L$ such that $\varphi' \circ k$ and ℓ are homotopic. Therefore we have the following commutative diagram except dotted arrow:

$$\begin{array}{ccccc} K & \xleftarrow{k} & M & & \\ \varphi' \downarrow & \searrow \varphi'' & \downarrow & \searrow \ell & \\ L & \xleftarrow{\sim \text{pr}_1} & \text{Path}(L) & \xrightarrow{\sim \text{pr}_2} & L \end{array}$$

where $\text{Path}(L)$ is a path object of L . Since k is a cofibration and pr_1 is a trivial fibration, there exists a morphism $\varphi'' : K \rightarrow \text{Path}(L)$ which completes the above commutative diagram. Now the composite morphism $\varphi := \text{pr}_2 \circ \varphi''$ satisfies the desired property. \square

• **Simplicial categories.** Recall that a simplicial category \mathcal{M} is a category which is enriched over the category \mathbf{sSet} of simplicial sets. For a simplicial category \mathcal{M} , we denote by $\mathrm{Map}_{\mathcal{M}}(K, L)$ the mapping complex for $K, L \in \mathcal{M}$. The underlying category \mathcal{M} of a simplicial category \mathcal{M} is a category the objects is the same with that of \mathcal{M} and the Hom set is defined to be $\mathrm{Map}_{\mathcal{M}}(K, L)_0$ the set of 0-simplexes of $\mathrm{Map}_{\mathcal{M}}(K, L)$. The homotopy category $\mathbf{h}\mathcal{M}$ of a simplicial category \mathcal{M} is a category the objects is the same with that of \mathcal{M} and the Hom set is defined to be $\mathrm{Hom}_{\mathbf{h}\mathcal{M}}(K, L) := \pi_0(\mathrm{Map}_{\mathcal{M}}(K, L))$. A simplicial functor $F : \mathcal{M} \rightarrow \mathcal{N}$ is called a *Dwyer-Kan equivalence* if the induced functor $\mathbf{h}F : \mathbf{h}\mathcal{M} \rightarrow \mathbf{h}\mathcal{N}$ gives an equivalence of categories and the induced map $\mathrm{Map}_{\mathcal{M}}(K, L) \rightarrow \mathrm{Map}_{\mathcal{N}}(F(K), F(L))$ is a weak equivalence for each pair $K, L \in \mathcal{M}$.

A simplicial category \mathcal{M} is called *fibrant* if every mapping complex $\mathrm{Map}_{\mathcal{M}}(K, L)$ is Kan complex, i.e., a fibrant object with respect to the standard model structure in \mathbf{sSet} . Let X be a simplicial set and $x \in X_0$ a 0-simplex of X . We often denote by $\{x\} \rightarrow X$ the map of simplicial set $\Delta^0 \rightarrow X$ corresponds to x under the natural bijection $X_0 \cong \mathrm{Map}_{\mathbf{sSet}}(\Delta^0, X)$. In particular for a morphism $\phi : K \rightarrow L$ we often denote by $\{\phi\} \rightarrow \mathrm{Map}_{\mathcal{M}}(K, L)$ the map of simplicial sets $\Delta^0 \rightarrow \mathrm{Map}_{\mathcal{M}}(K, L)$ which corresponds to the 0-simplex $\phi \in \mathrm{Map}_{\mathcal{M}}(K, L)_0 = \mathrm{Hom}_{\mathcal{M}}(K, L)$.

Let M be an object of \mathcal{M} . We equip the under category $\mathcal{M}_{M/}$ with a structure of simplicial category in the following way: let $k : M \rightarrow K$ and $\ell : M \rightarrow L$ be objects of \mathcal{M} . Then the mapping complex $\mathrm{Map}_{\mathcal{M}_{M/}}(\ell, k)$ is defined to be the sub simplicial set of $\mathrm{Map}_{\mathcal{M}}(L, K)$ which fits into the following pull back diagram of simplicial sets

$$(2-1) \quad \begin{array}{ccc} \mathrm{Map}_{\mathcal{M}_{M/}}(\ell, k) & \hookrightarrow & \mathrm{Map}_{\mathcal{M}}(L, K) \\ \downarrow & & \downarrow \ell^* \\ \{k\} & \longrightarrow & \mathrm{Map}_{\mathcal{M}}(M, K). \end{array}$$

where the right vertical arrow $\ell^* := \mathrm{Map}_{\mathcal{M}}(\ell, K)$ is the induced morphism.

In a similar way, we equip the over category $\mathcal{M}_{/M}$ with a structure of simplicial category.

A functor $- \otimes - : \mathcal{M} \times \mathbf{sSet} \rightarrow \mathcal{M}$ is called *tensor product* if we have a natural isomorphism of simplicial sets

$$\mathrm{Map}_{\mathbf{sSet}}(X, \mathrm{Map}_{\mathcal{M}}(K, L)) \cong \mathrm{Map}_{\mathcal{M}}(K \otimes X, L)$$

for $X \in \mathbf{sSet}$ and $K, L \in \mathcal{M}$. A functor $\mathcal{M} \times \mathbf{sSet}^{\mathrm{op}} \rightarrow \mathcal{M}$, $(K, X) \mapsto K^X$ is called *cotensor product* if we have a natural isomorphism of simplicial sets

$$\mathrm{Map}_{\mathbf{sSet}}(X, \mathrm{Map}_{\mathcal{M}}(K, L)) \cong \mathrm{Map}_{\mathcal{M}}(K, L^X).$$

Let \mathcal{M} be a simplicial category having a cotensor product $\mathcal{M} \times \mathbf{sSet}^{\mathrm{op}} \rightarrow \mathcal{M}$. For an object $k : M \rightarrow K$ of the under category $\mathcal{M}_{M/}$ and a simplicial set X , we define an object k^X of $\mathcal{M}_{/M}$ to be the object given by the composition

$$M \xrightarrow{k} K \cong K^{\Delta^0} \xrightarrow{K^{\mathbf{uni}}} K^X$$

where we denote by \mathbf{uni} a unique map $X \rightarrow \Delta^0$. Then we claim that the functor $\mathcal{M}_{M/} \times \mathbf{sSet}^{\mathrm{op}} \rightarrow \mathcal{M}$, $(k, X) \mapsto k^X$ is a cotensor product. Indeed we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathbf{sSet}}(X, \mathrm{Map}_{\mathcal{M}}(L, K)) & \xlongequal{\sim} & \mathrm{Map}_{\mathcal{M}}(L, K^X) \\ \mathrm{Map}_{\mathbf{sSet}}(X, \ell^*) \downarrow & & \downarrow \ell^* \\ \mathrm{Map}_{\mathbf{sSet}}(X, \mathrm{Map}_{\mathcal{M}}(M, K)) & \xlongequal{\sim} & \mathrm{Map}_{\mathcal{M}}(M, K^X) \\ \uparrow & & \uparrow \\ \mathrm{Map}_{\mathbf{sSet}}(X, \{k\}) & \xlongequal{\sim} & \{k^X\}, \end{array}$$

which implies that we have a natural isomorphism

$$\mathrm{Map}_{\mathbf{sSet}}(X, \mathrm{Map}_{\mathcal{M}_{M/}}(\ell, k)) \cong \mathrm{Map}_{\mathcal{M}_{M/}}(\ell, k^X).$$

In a similar way, for a simplicial category \mathcal{M} having a tensor product $- \otimes - : \mathcal{M} \times \mathbf{sSet} \rightarrow \mathcal{M}$, we equip the over category $\mathcal{M}_{/M}$ with a tensor product in the following way: for an object $p : P \rightarrow M$ of $\mathcal{M}_{/M}$ and a simplicial set X we define an object $p \otimes X$ of $\mathcal{M}_{/M}$ to be the object given by the composition

$$P \otimes X \xrightarrow{1_P \otimes \mathbf{uni}} P \otimes \Delta^0 \cong P \xrightarrow{p} M$$

where we denote by \mathbf{uni} a unique map $X \rightarrow \Delta^0$.

• **Simplicial model categories.** We recall that a simplicial model category \mathcal{M} is not only a simplicial category equipped with a model structure, but also assumed to have a tensor product and a cotensor product satisfying axioms which state that the structure of simplicial category and the model structure are consistent. We reminded the readers that

1. for a cofibration $K \rightarrowtail K'$ and a fibrant object L of a simplicial model category \mathcal{M} , the induced morphism $\mathrm{Map}_{\mathcal{M}}(K', L) \rightarrow \mathrm{Map}_{\mathcal{M}}(K, L)$ is a fibration of simplicial sets
2. for a cofibrant object K and a fibration $L \rightarrow L'$, the induced morphism $\mathrm{Map}_{\mathcal{M}}(K, L) \rightarrow \mathrm{Map}_{\mathcal{M}}(K, L')$ is a fibration.
3. the fibrant-cofibrant part \mathcal{M}° is a fibrant simplicial category.
4. the homotopy category $\mathrm{Ho}(\mathcal{M}^\circ)$ with respect to the model structure and the homotopy category $\mathrm{h}\mathcal{M}^\circ$ with respect to the structure of simplicial category are equivalent. In other words the identity on the objects gives an equivalence $\mathrm{Ho}(\mathcal{M}^\circ) \simeq \mathrm{h}\mathcal{M}^\circ$.

In particular, the last two properties are important in a relationship with ∞ -categories.

The under category $\mathcal{M}_{M/}$ of a simplicial model category \mathcal{M} fail to become a simplicial category, since $\mathcal{M}_{M/}$ is not tensored over \mathbf{sSet} . However we can prove

Lemma 2.2. *Let $k : M \rightarrow K$ be a fibrant object and $\ell : M \rightarrow L$ a cofibrant object of the under category $\mathcal{M}_{M/}$.*

(1) *The mapping complex $\mathrm{Map}_{\mathcal{M}_{M/}}(\ell, k)$ is Kan complex. In particular the simplicial category \mathcal{M}° is a fibrant simplicial category.*

(2) *The object k^{Δ^1} is a path object of p .*

(3) *Two morphisms $f, g : \ell \rightarrow k$ are homotopic if and only if then they are weak homotopy equivalence, i.e., the images of f, g in $\pi_0(\mathrm{Map}_{\mathcal{M}_{M/}}(\ell, k))$ coincide.*

(4) *Assume that k and ℓ are fibrant-cofibrant objects. Then the morphism $\phi : \ell \rightarrow k$ is weak homotopy equivalence if and only if ϕ is weak equivalence. In particular the identity on the objects gives an equivalence $\mathrm{h}\mathcal{M}_{M/}^\circ \simeq \mathrm{Ho}(\mathcal{M}_{M/}^\circ)$.*

Proof. (1) By the definition of the model structure in $\mathcal{M}_{M/}$, the morphism $\ell : M \rightarrow L$ is a cofibration in \mathcal{M} and the co-domain K of k is a fibrant object of \mathcal{M} . Hence the induced morphism $\ell^* = \mathrm{Map}_{\mathcal{M}}(\ell, K) : \mathrm{Map}_{\mathcal{M}}(L, K) \rightarrow \mathrm{Map}_{\mathcal{M}}(M, K)$ is a Kan fibration of simplicial sets. By the pull back diagram (2-1) we conclude that the simplicial set $\mathrm{Map}_{\mathcal{M}_{M/}}(\ell, k)$ is a Kan complex.

(2) Using the characterization of fibrations and cofibrations of $\mathcal{M}_{/M}$ and the axioms of simplicial model category, we can check that the morphism $k^{\mathbf{uni}} : k \rightarrow k^{\Delta^1}$ is weak equivalence and the induced morphism $k^\iota : k^{\Delta^1} \rightarrow k^{\{0\} \sqcup \{1\}} \cong k \times k$ is a fibration where $\iota : \{0\} \sqcup \{1\} \rightarrow \Delta^1$ is the canonical inclusion.

(3) Since the mapping complex $\text{Map}_{\mathcal{M}_{M/}}(\ell, k)$ is a Kan complex by (1), the 0-th homotopy is given by the co-equalizer

$$\pi_0(\text{Map}_{\mathcal{M}_{M/}}(\ell, k)) = \text{coeq} \left[d_0, d_1 : \text{Map}_{\mathcal{M}_{M/}}(\ell, k)_1 \rightrightarrows \text{Map}_{\mathcal{M}_{M/}}(\ell, k)_0 \right]$$

where d_0, d_1 are the boundary maps. Let $\iota_0 : \Delta^0 \cong \{0\} \rightarrow \Delta^1$ and $\iota_1 : \Delta^0 \cong \{1\} \rightarrow \Delta^1$ be canonical inclusions. We have the following commutative diagram

$$\begin{array}{ccc} \text{Map}_{\mathcal{M}_{M/}}(\ell, k)_1 & \xlongequal{\sim} & \text{Hom}_{\mathcal{M}_{M/}}(\ell, k^{\Delta^1}) \\ d_a \downarrow & & \downarrow \text{Hom}_{\mathcal{M}_{M/}}(\ell, k^{\iota_a}) \\ \text{Map}_{\mathcal{M}_{M/}}(\ell, k)_0 & \xlongequal{\sim} & \text{Hom}_{\mathcal{M}_{M/}}(\ell, k) \end{array}$$

for $a = 0, 1$. Therefore two morphisms $f, g : \ell \rightarrow k$ are weak homotopy equivalent if and only if we have a commutative diagram

$$(2-2) \quad \begin{array}{ccccc} & & \ell & & \\ & f \swarrow & \downarrow H & \searrow g & \\ k & \xleftarrow{k^{\iota_0}} & k^{\Delta^1} & \xrightarrow{k^{\iota_1}} & k. \end{array}$$

for some $H : \ell \rightarrow k^{\Delta^1}$. Thus by (2) if f and g are weak homotopy equivalent then they are homotopic.

Conversely assume that f and g are homotopic. Then by [9, Proposition 1.2.5] there exists a right homotopy from f to g using the path object k^{Δ^1} . Namely we have the above commutative diagram (2-2) for some H . Therefore f and g are weak homotopy equivalent.

(4) follows from (3). □

In a similar way we can prove

Lemma 2.3. *Let $p : P \rightarrow M$ be a fibrant object and $q : Q \rightarrow M$ a cofibrant of $\mathcal{M}_{/M}$.*

(1) *The mapping complex $\text{Map}_{\mathcal{M}_{/M}}(p, q)$ is a Kan complex. In particular the simplicial category $\mathcal{M}_{/M}^\circ$ is a fibrant simplicial category.*

(2) *The object $q \otimes \Delta^1$ is a cylinder object of q .*

(3) *Two morphisms $f, g : p \rightarrow q$ are homotopic if and only if they are weak homotopy equivalence, i.e., the images of f, g in $\pi_0(\text{Map}_{\mathcal{M}_{/M}}(p, q))$ coincide.*

(4) *Assume that p and q are fibrant-cofibrant. Then the morphism $\phi : p \rightarrow q$ is weak equivalence if and only if ϕ is weak homotopy equivalence. In particular, the identity on the objects gives an equivalence $\text{h}\mathcal{M}_{/M}^\circ \simeq \text{Ho}(\mathcal{M}_{/M}^\circ)$*

2.2 Homotopy theory of dg-categories

For dg-categories, we refer to [4, 12, 13, 34, 36, 37].

• The category $\mathcal{C}(\mathcal{A})$ of dg \mathcal{A} -modules.

Trough out of this paper \mathbf{k} is a commutative ring and dg-categories are dg-categories over \mathbf{k} . Let \mathcal{A} be a small dg-category. For a pair of objects $a, b \in \mathcal{A}$, we denote by $\text{Hom}_{\mathcal{A}}^\bullet(a, b)$ the Hom complex of \mathcal{A} . We denote by $\mathcal{C}(\mathcal{A})$ the category of dg \mathcal{A} -modules and by $\mathcal{C}_{\text{dg}}(\mathcal{A})$ the dg-category of dg \mathcal{A} -modules. For simplicity we set ${}_{\mathcal{A}}(M, N) := \text{Hom}_{\mathcal{C}_{\text{dg}}(\mathcal{A})}^\bullet(M, N)$ for dg \mathcal{A} -modules M and N .

It is known that $\mathcal{C}(\mathcal{A})$ admits two structure if combinatorial model category whose weak equivalences are the quasi-isomorphisms:

1. The projective model structure, whose fibrations are the term-wise surjection.
2. The injective model structure, whose cofibrations are the term-wise injection.

Note that every object of $\mathcal{C}(\mathcal{A})$ is projective fibrant and injective cofibrant. Through out of this paper we equip $\mathcal{C}(\mathbf{k})$ with the projective model structure.

For an object $a \in \mathcal{A}$, we denote by a^\wedge the free dg \mathcal{A} -module at a , that is, the dg-functor $\mathcal{A}^{\text{op}} \rightarrow \mathcal{C}_{\text{dg}}(\mathbf{k})$ which sends $b \in \mathcal{A}$ to $\mathcal{A}^\bullet(b, a)$. A dg \mathcal{A} -module Q is called *relative projective* if, in $\mathcal{C}(\mathcal{A})$, it is a direct summand of a direct sum of dg-modules of the form $a^\wedge[n]$, $a \in \mathcal{A}, n \in \mathbb{Z}$. We denote by ${}^\wedge a$ the free dg left \mathcal{A} -module at a , that is, the dg-functor $\mathcal{A} \rightarrow \mathcal{C}_{\text{dg}}(\mathbf{k})$ which sends $b \in \mathcal{A}$ to $\mathcal{A}^\bullet(a, b)$.

We fix an injective co-generator E of $\text{Mod } \mathbf{k}$. We set $\mathbf{D} := {}_{\mathbf{k}}(-, E)$. We define $a^\vee := \mathbf{D}({}^\wedge a)$. Namely a^\vee is the dg \mathcal{A} -module $\mathcal{A}^{\text{op}} \rightarrow \mathcal{C}(\mathbf{k})$ which sends $b \in \mathcal{A}$ to ${}_{\mathbf{k}}(\mathcal{A}^\bullet(a, b), E)$. A dg \mathcal{A} -module is called *relative injective* if, in $\mathcal{C}(\mathcal{A})$, it is a direct summand of a direct product of modules of the form $a^\vee[n]$, $a \in \mathcal{A}, n \in \mathbb{Z}$. We can easily check that if N is a projectively cofibrant left dg \mathcal{A} -module, then $\mathbf{D}(N)$ is injectively fibrant.

Let $\mathcal{A}^\#$ be the underlying graded category of \mathcal{A} . We denote by $\mathcal{G}(\mathcal{A}^\#)$ the category of graded $\mathcal{A}^\#$ modules. Let $\# : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{G}(\mathcal{A}^\#)$ be the forgetful functor. A dg \mathcal{A} -module M is determined by $M^\#$ and the differential d_M on $M^\#$. Therefore we may write $M = (M^\#, d_M)$. By abusing notation, we often write $M = (M, d_M)$. A sequence $L \rightarrow M \rightarrow N$ of dg \mathcal{A} -modules is called *#-exact* if the induced sequence $L^\# \rightarrow M^\# \rightarrow N^\#$ is an exact sequence of graded $\mathcal{A}^\#$ -modules. Since the functor $\#$ is faithful, we may use the morphism $f^\# : M^\# \rightarrow N^\#$ of the underlying modules to denote a morphism $f : M \rightarrow N$ of dg \mathcal{A} -modules.

For a morphism $f : M \rightarrow N$, we set the cone $c(f)$ and the co-cone $cc(f)$ to be

$$\begin{aligned} c(f) &:= \left(N \oplus M[1], \begin{pmatrix} d_N & f \\ 0 & -d_M \end{pmatrix} \right), \\ cc(f) &:= \left(N[-1] \oplus M, \begin{pmatrix} -d_N & f \\ 0 & d_M \end{pmatrix} \right). \end{aligned}$$

Observe that there are following #-exact sequences

$$\begin{aligned} (2-3) \quad 0 &\rightarrow N \xrightarrow{t(1_N, 0)} c(f) \xrightarrow{(0, 1_{M[1]})} M[1] \rightarrow 0, \\ 0 &\rightarrow N[-1] \xrightarrow{t(1_{N[-1]}, 0)} cc(f) \xrightarrow{(0, 1_M)} M \rightarrow 0. \end{aligned}$$

By [12, Proof of Lemma 2.3] the functor $\#$ has the left adjoint F_λ and the right adjoint F_ρ . Therefore limits and colimits commute with $\#$. In other words, the underlying graded module of a (co)limit dg \mathcal{A} -module is obtained as the (co)limit of the underlying graded modules.

A dg \mathcal{A} -module M is called a *free* if it is a direct sum of modules of the form $a^\wedge[n]$ for $a \in \mathcal{A}$ and $n \in \mathbb{Z}$. A dg \mathcal{A} -module P is called a *semi-free module* if it has an exhaustive filtration $0 = P_0 \subset P_1 \subset \cdots \subset P$ such that each quotient P_i/P_{i-1} is free. Note that the underlying graded $\mathcal{A}^\#$ -module $P^\#$ of a semi-free dg \mathcal{A} -module P is free and that a semi-free dg \mathcal{A} -module P is projectively cofibrant. It is known that every dg \mathcal{A} -module M has a semi-free resolution, i.e., a projectively trivial fibration $P \xrightarrow{\sim} M$ with P semi-free. (See [4, Appendix III].) Dually let Q be a semi-free left dg \mathcal{A} -module. Then the dg \mathcal{A} -module $\mathbf{D}(Q)$ is injectively fibrant and the underlying graded $\mathcal{A}^\#$ -module $\mathbf{D}(Q)^\#$ is injective. Moreover for every dg \mathcal{A} -module M there exists a trivial injective cofibration $M \xrightarrow{\sim} \mathbf{D}(Q)$ for some semi-free left module Q .

Lemma 2.4. *Every projective cofibrant object M is a direct summand of some semi-free module P . Consequently the underlying graded $\mathcal{A}^\#$ -module $M^\#$ is projective. Dually every injective fibrant*

object N is a direct summand of $\mathbf{D}(Q)$ for some left semi-free module Q . Consequently the graded $\mathcal{A}^\#$ -module $N^\#$ is injective.

Proof. We prove the first statement. The second statement is proved in a dual way. Let $p : P \xrightarrow{\sim} M$ be a semi-free resolution. Using the lifting property, we see that p split and M is a direct summand of P . \square

Lemma 2.5. *Let $f : M \rightarrow N$ be a morphism of dg \mathcal{A} -modules.*

(1) *f is a projective cofibration if and only if it is a term-wise injection with the projective cofibrant cokernel.*

(2) *f is an injective fibration if and only if it is a term-wise surjection with the injective fibrant kernel.*

Proof. Using Lemma 2.4 we can prove this lemma by the same methods of [9, Lemma 2.3.9, Lemma 2.3.20]. \square

Lemma 2.6. (1) *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a $\#$ -exact sequence of dg \mathcal{A} -modules. Assume that L and N are projectively cofibrant. Then so is the middle term M . Dually, if L and N are injectively fibrant, then so is M .*

(2) *Let $f : M \rightarrow N$ be a morphism of dg \mathcal{A} -modules. If M and N are injectively fibrant, then the cone $c(f)$ and the co-cone $cc(f)$ are injectively fibrant. Dually, if M and N are projectively cofibrant, then the cone and the co-cone are projectively cofibrant.*

Proof. We prove the first statements of (1) and (2). The second statements are proved in dual ways. (1) By Lemma 2.5 the morphism $L \rightarrow M$ is projective cofibration. Since it is a cofibration from a cofibrant object, the object M is projectively cofibrant object.

(2) It is clear that the shift functor $[-1]$ preserves projectively cofibrant object. Since $cc(f) = c(f)[-1]$, it is enough to check that $c(f)$ is projective cofibrant. From the $\#$ -exact sequence (2-3) we conclude that $c(f)$ is projectively cofibrant. \square

• **The category \mathbf{dgCat} of dg-categories.** We denote by $H^0(\mathcal{A})$ the homotopy category of the dg-category \mathcal{A} . Namely this is a graded category such that the objects of $H^0(\mathcal{A})$ is the same with that of \mathcal{A} and the Hom module is given by $H^0(\mathcal{A})(a, b) := H^0(\mathcal{A}^\bullet(a, b))$. A dg-functor $f : \mathcal{A} \rightarrow \mathcal{B}$ is called a *quasi-equivalence* if the induced functor $H^0(f) : H^0(\mathcal{A}) \rightarrow H^0(\mathcal{B})$ gives an equivalence of categories and the induced map $\mathrm{Hom}_{\mathcal{A}}^\bullet(a, b) \rightarrow \mathrm{Hom}_{\mathcal{B}}^\bullet(f(a), f(b))$ is a quasi-isomorphism.

Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a dg-functor. We have an adjoint pair

$$(2-4) \quad f^* := - \otimes_{\mathcal{A}} \mathcal{B} : \mathcal{C}(\mathcal{A}) \rightleftarrows \mathcal{C}(\mathcal{B}) : f_*$$

where f_* is the restriction functor. Recall that the restriction functor f_* is defined to be $(f_*M)(a) := M(fa)$ for $a \in \mathcal{A}$ and $M \in \mathcal{C}(\mathcal{B})$. Therefore f_* preserves quasi-isomorphisms, injective cofibrations and projective cofibrations. Note that we have a canonical isomorphism $f^*(a^\wedge) \cong (fa)^\wedge$ for $a \in \mathcal{A}$.

Lemma 2.7. *If we equip both $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{B})$ with the projective model structures. Then the above adjoint (2-4) become a Quillen adjoint pair. Moreover if f is quasi-equivalence, then the above adjoint pair (2-4) gives a Quillen equivalence.*

The tensor product $\mathcal{A} \otimes \mathcal{B}$ of dg-categories \mathcal{A} and \mathcal{B} is defined in the following way. The sets of object is given by the product $\mathrm{Ob}(\mathcal{A} \otimes \mathcal{B}) = \mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{B})$. We denote by $a \otimes b$ the object of $\mathcal{A} \otimes \mathcal{B}$ corresponding to $(a, b) \in \mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{B})$. The Hom-complex is given by $(\mathcal{A} \otimes \mathcal{B})^\bullet(a \otimes b, a' \otimes b') := \mathcal{A}^\bullet(a, a') \otimes \mathcal{B}^\bullet(b, b')$. The composition and the identities are defined in an obvious way.

Lemma 2.8. *Let \mathcal{A} be a locally cofibrant dg-category and $f : \mathcal{B} \rightarrow \mathcal{C}$ a quasi-equivalence. Then the induced functor $1 \otimes f : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A} \otimes \mathcal{C}$ is a quasi-equivalence.*

We denote by dgCat the category of small dg-categories. It is proved by Tabuada [34] that the category dgCat admits a structure of cofibrantly generated model category whose weak equivalences are the quasi-equivalences. (The characterization of fibration is known. However we don't use it.)

A dg-category \mathcal{A} is called *locally cofibrant* if for $a, b \in \mathcal{A}$ the Hom complex $\text{Hom}_{\mathcal{A}}^{\bullet}(a, b)$ is projective cofibrant in $\mathcal{C}(\mathbf{k})$. Note that by [36, Lemma 2.3] cofibrant dg-category is locally cofibrant. Using the same method of [loc. cite], we can prove the following lemma.

Lemma 2.9. *Let \mathcal{A} be a locally cofibrant small dg-category and $f : \mathcal{A} \rightarrow \mathcal{B}$ a cofibration in dgCat . Then for any $b \in \mathcal{B}$, the dg \mathcal{A} -module $f_*(b^{\wedge})$ is projective cofibrant. Dually for any $b \in \mathcal{B}$, the left dg \mathcal{A} module $f_*(^{\wedge}b)$ is projective cofibrant left dg \mathcal{A} -module.*

Note that if $f' : \mathcal{A} \rightarrow \mathcal{B}'$ be a dg-functor such that left dg \mathcal{A} -module $f'_*(^{\wedge}b)$ is projective cofibrant for $b \in \mathcal{B}'$. Then for any full dg-subcategory $\mathcal{B} \subset \mathcal{B}'$ which contains the image of f' , the factorization $f : \mathcal{A} \rightarrow \mathcal{B}$ of $f' : \mathcal{A} \rightarrow \mathcal{B}'$ has the same property.

Lemma 2.10. *Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a morphism in dgCat such that left dg \mathcal{A} -module $f_*(^{\wedge}b)$ is projective cofibrant for $b \in \mathcal{B}$. (e.g. the essential image of some cofibration $\mathcal{A} \rightarrow \mathcal{B}'$ in dgCat .) Then the adjoint pair induced from f*

$$f^* : \mathcal{C}(\mathcal{A}) \rightleftarrows \mathcal{C}(\mathcal{B}) : f_*$$

is a Quillen adjoint where both $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{B})$ are equipped with the injective model structure.

Proof. We prove that f_* preserves injective (trivial) fibrations. It is clear that the restriction functor f_* preserves term-wise surjections and weakly contractibility. Therefore, in view of Lemma 2.5, it is enough to check that f_* preserves injective fibrant objects. By Lemma 2.4 it is enough to check that $f_*\mathbf{D}(Q)$ is injectively fibrant for each semi-free left module Q . Since the functor f_* commutes with limit and the dual \mathbf{D} sends colimit to limit, we have only to check the case when $Q = ^{\wedge}b$ for some $b \in \mathcal{B}$, which follows from the assumption and the fact that f_* and \mathbf{D} are commute. \square

Lemma 2.11. *Let \mathcal{A} be a dg-category, \mathcal{B} locally cofibrant dg-category and J an injectively fibrant dg $\mathcal{A} \otimes \mathcal{B}$ -modules. Then, for any $b \in \mathcal{B}$, the \mathcal{A} -module $J(- \otimes b)$ is injectively fibrant.*

Proof. We prove the first statement. We can prove the second statement in the same way. We denote by η the dg-functor $\mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B}$, $a \mapsto a \otimes b$. Then the induced adjoint pair $\eta^* : \mathcal{C}(\mathcal{A}) \rightleftarrows \mathcal{C}(\mathcal{A} \otimes \mathcal{B}) : \eta_*$ is given by $\eta^* : N \mapsto N \otimes b^{\wedge}$ and $\eta_* : X \mapsto X(- \otimes b)$. Since we assume that the dg-category \mathcal{B} is locally cofibrant, the functor η^* preserves injective cofibration and trivial injective cofibration. Hence the adjoint pair $\eta^* \dashv \eta_*$ is a Quillen pair when we equip the both categories $\mathcal{C}(\mathcal{A}), \mathcal{C}(\mathcal{A} \otimes \mathcal{B})$ with the injective model structures. In particular the η_* functor preserves injectively fibrant objects. Therefore $J(- \otimes b)$ is injectively fibrant. \square

• **The simplicial model category $\mathcal{C}_{\text{ss}}(\mathcal{A})$ of dg \mathcal{A} -modules.** Through out this paper we equip the category $\mathcal{C}(\mathbf{k})$ with the projective model structure. Then $\mathcal{C}(\mathbf{k})$ with a canonical tensor product become a monoidal model category, and the category $\mathcal{C}(\mathcal{A})$ either with the projective model structure or with the injective model structure become a $\mathcal{C}(\mathbf{k})$ -enriched model category in the sense of [20, Definition A. 3.1.5]. The following lemma is deduced from this fact.

Lemma 2.12. (1) *Let M be a dg \mathcal{A} -module. We equip $\mathcal{C}(\mathcal{A})$ with the injective model structure. Then the tensor-Hom adjunction*

$$- \otimes M : \mathcal{C}(\mathbf{k}) \rightleftarrows \mathcal{C}(\mathcal{A}) : {}_{\mathcal{A}}(M, -)$$

is a Quillen adjunction. In particular, the functor $\mathcal{A}(M, -)$ preserves weak equivalences between injectively fibrant objects.

(2) Let M be a projectively cofibrant dg \mathcal{A} -module. We equip $\mathcal{C}(\mathcal{A})$ with the projective model structure. Then the tensor-Hom adjunction

$$- \otimes M : \mathcal{C}(\mathbf{k}) \rightleftarrows \mathcal{C}(\mathcal{A}) : \mathcal{A}(M, -)$$

is a Quillen adjunction. In particular, the functor $\mathcal{A}(M, -)$ preserves weak equivalences.

(3) The Hom complex $\mathcal{A}(M, N)$ is a cofibrant object of $\mathcal{C}(\mathbf{k})$, if M is projectively cofibrant or N is injectively fibrant.

We denote by $\text{Ch}_{\geq 0}(\mathbf{k})$ the category of non-negatively graded chain complexes of \mathbf{k} -modules. We denote by $\text{sMod } \mathbf{k}$ the category of simplicial \mathbf{k} -modules. Both categories have a canonical monoidal structure. Then there is a monoidal functor $\Gamma : \text{Ch}_{\geq 0}(\mathbf{k}) \rightarrow \text{sMod } \mathbf{k}$ which gives an equivalence of the underlying categories. (See [31].)

We denote by $\mathcal{C}_{\text{ss}}(\mathcal{A})$ the associated simplicial category of the dg-category $\mathcal{C}_{\text{dg}}(\mathcal{A})$. Namely, the simplicial category $\mathcal{C}_{\text{ss}}(\mathcal{A})$ is a simplicial category objects of which are the same with $\mathcal{C}(\mathcal{A})$ and the mapping complex which is denoted by $\text{Map}_{\mathcal{A}}$ is given by

$$\text{Map}_{\mathcal{A}}(M, N) := \Gamma \tilde{\tau}^{\leq 0}(\mathcal{A}(M, N))$$

where $\tilde{\tau}^{\leq 0}$ is the composition of the truncation functor $\tau^{\leq 0} : \mathcal{C}(\mathbf{k}) \rightarrow \mathcal{C}_{\leq 0}(\mathbf{k})$ with a canonical equivalence $(\tilde{-}) : \mathcal{C}^{\leq 0}(\mathbf{k}) \xrightarrow{\sim} \text{Ch}_{\geq 0}(\mathbf{k})$. Note that for a complex $L \in \mathcal{C}(\mathbf{k})$, we have $\Gamma \tilde{\tau}^{\leq 0} L = Z^0(L)$. Therefore the underlying category of the simplicial category $\mathcal{C}_{\text{ss}}(\mathcal{A})$ is the category $\mathcal{C}(\mathcal{A})$.

The simplicial category $\mathcal{C}_{\text{ss}}(\mathcal{A})$ is tensored and cotensored over sSet in the following way. For a simplicial set X , we denote by $\mathbf{k}[X]$ the free \mathbf{k} -module generated by X . Let $\tilde{N} : \text{sMod } \mathbf{k} \rightarrow \text{Ch}_{\geq 0}$ be the composition of the normalization functor $N : \text{sMod } \mathbf{k} \rightarrow \text{Ch}_{\geq 0} \mathbf{k}$ with a canonical equivalence $\text{Ch}_{\geq 0}(\mathbf{k}) \xrightarrow{\sim} \mathcal{C}^{\leq 0}(\mathbf{k})$. Then for $M \in \mathcal{C}(\mathcal{A})$ and $X \in \text{sSet}$, the tensor $M \otimes X$ and the cotensor M^X is given by

$$M \otimes X := M \otimes \tilde{N}(\mathbf{k}[X]), \quad M^X := M^{\tilde{N}(\mathbf{k}[X])}.$$

We have a natural isomorphism of simplicial sets

$$\text{Map}_{\text{sSet}}(X, \text{Map}_{\mathcal{A}}(M, N)) \cong \text{Map}_{\mathcal{A}}(M \otimes X, N) \cong \text{Map}_{\mathcal{A}}(M, N^X).$$

Note that since every simplicial set X is cofibrant, the tensor product $M \otimes X$ preserves projectively cofibrant objects and the cotensor product M^X preserves injectively fibrant objects.

Let X be a finite simplicial set. Then the corresponding dg \mathbf{k} -module $\tilde{N}(\mathbf{k}[X])$ is a strictly perfect complex, i.e., a bounded complex each term of which is finite projective \mathbf{k} -module. Therefore we can construct the tensor product $M \otimes X$ and the cotensor product M^X from M by taking shift, cone and direct summands.

The complexes $\mathbf{k} \otimes \Delta^0 = \tilde{N}(\mathbf{k}[\Delta^0])$ and $\mathbf{k} \otimes \Delta^1 = \tilde{N}(\mathbf{k}[\Delta^1])$ which correspond to the standard 0-simplex and the standard 1-simplex are given by

$$\tilde{N}(\mathbf{k}[\Delta^0]) = \mathbf{k}, \quad \tilde{N}(\mathbf{k}[\Delta^1]) = \left(\mathbf{k} \oplus \mathbf{k} \oplus \mathbf{k}[1], \begin{pmatrix} 0 & 0 & 1_{\mathbf{k}} \\ 0 & 0 & -1_{\mathbf{k}} \\ 0 & 0 & 0 \end{pmatrix} \right).$$

where we consider \mathbf{k} as the complex concentrated at 0-th degree. We denote by ι_a the canonical inclusion $\Delta^0 \cong \{a\} \rightarrow \Delta^1$ for $a = 0, 1$ and by \mathbf{uni} a unique map $\Delta^1 \rightarrow \Delta^0$. Then the corresponding maps are given by

$$\begin{aligned} \mathbf{k} \otimes \iota_0 &= {}^t(0, 1_{\mathbf{k}}, 0), \mathbf{k} \otimes \iota_1 = {}^t(1_{\mathbf{k}}, 0, 0) : \mathbf{k} \otimes \Delta^0 \rightarrow \mathbf{k} \otimes \Delta^1 \\ \mathbf{k} \otimes \mathbf{uni} &= (1_{\mathbf{k}}, 1_{\mathbf{k}}, 0) : \mathbf{k} \otimes \Delta^1 \rightarrow \mathbf{k} \otimes \Delta^0 \end{aligned}$$

Lemma 2.13. *Let M be a dg \mathcal{A} -module. Assume that in the under category $\mathcal{C}(\mathcal{A})_{M/}$ two morphism $s_1 : k_1 \rightarrow k_0$ and $s_2 : k_2 \rightarrow k_0$ are given. Then there exist morphisms $t_1 : k_3 \rightarrow k_1$ and $t_2 : k_3 \rightarrow k_2$ such that $s_1 \circ t_1$ and $s_2 \circ t_2$ are weak homotopy equivalent. In other words, any diagram $k_1 \rightarrow k_0 \leftarrow k_2$ in $\mathcal{C}(\mathcal{A})_{M/}$ can be completed in a homotopy commutative diagram*

$$\begin{array}{ccc} k_3 & \longrightarrow & k_2 \\ \downarrow & & \downarrow \\ k_1 & \longrightarrow & k_0 \end{array}$$

Dually for morphisms $s_1 : p_0 \rightarrow p_1$ and $s_2 : p_0 \rightarrow p_2$ in the over category $\mathcal{C}(\mathcal{A})_{/M}$ there exist a morphism $t_1 : p_1 \rightarrow p_3$ and $t_2 : p_2 \rightarrow p_3$ such that $t_1 \circ s_1$ and $t_2 \circ s_2$ are weak homotopy equivalent.

Proof. We prove the first statement. The second statement is proved in a dual way.

We write $k_i : R \rightarrow K_i$ for $i = 0, 1, 2$. We denote by the same symbol s_i the morphisms $K_i \rightarrow K_0$ between co-domains induced from $s_i : k_i \rightarrow k_0$ for $i = 1, 2$. We define K_3 to be the co-cone $\text{cc}(-s_1, s_2)$ of the morphism $(-s_1, s_2) : K_1 \oplus K_2 \rightarrow K_0$. We define a morphism $k_3 : M \rightarrow K_3$ of graded modules to be $k_3 := {}^t(0, k_1, k_2) : R \rightarrow K_3 = K_0[-1] \oplus K_1 \oplus K_2$. Using the calculation $(s_1, -s_2) \circ {}^t(k_1, k_2) = 0$, we check that k_3 is a chain map. We denote by $t_i : k_3 \rightarrow k_i$ the morphism in \mathcal{U} induced from the $i + 1$ -th projection $\text{pr}_{i+1} : K_3 = K_0[-1] \oplus K_1 \oplus K_2 \rightarrow K_i$ for $i = 1, 2$. Observe that

$$K_0^{\Delta^1} = \left(K_0[-1] \oplus K_0 \oplus K_0, \begin{pmatrix} -d_{K_0} & -1_{K_0} & 1_{K_0} \\ 0 & d_{K_0} & 0 \\ 0 & 0 & d_{K_0} \end{pmatrix} \right).$$

Recall that the cotensor $k_0^{\Delta^1}$ is defined to be $M \xrightarrow{k_0} K_0 \xrightarrow{K_0^{\text{uni}}} K_0^{\Delta^1}$. We denote by H the morphism $\text{diag}(1_{K_0[-1]}, s_1, s_2) : K_3 \rightarrow K_0^{\Delta^1}$. Then it is a straightforward calculation to check the equation $k_0^{\Delta^1} = H \circ k_3$ of morphisms in $\mathcal{C}(\mathcal{A})$. In other words, the morphism H induces a morphism $k_3 \rightarrow k_0$ in the under category $\mathcal{C}(\mathcal{A})_{M/}$, which will be also denoted by H .

Now we can check that the following diagram in \mathcal{U} is commutative:

$$\begin{array}{ccccc} k_1 & \xleftarrow{t_1} & k_3 & \xrightarrow{t_2} & k_2 \\ s_1 \downarrow & & \downarrow H & & \downarrow s_2 \\ k_0 & \xleftarrow{k_0^{\iota_0}} & k_0^{\Delta^1} & \xrightarrow{k_0^{\iota_1}} & k_0 \end{array}$$

By Lemma 2.2 this diagram gives the desired homotopy commutative diagram. \square

Remark 2.14. From the construction of K_3 in the above proof, we see the following things: If the co-domains of k_0, k_1, k_2 are injectively fibrant, we can take k_3 whose co-domain is also injectively fibrant. if one of the morphism $k_a : M \rightarrow K_a$ ($a=1,2$) is injectively cofibration, then we can take an object $k_3 \in \mathcal{C}(\mathcal{A})_{M/}$ such that the corresponding morphism $k_3 : M \rightarrow K_3$ in $\mathcal{C}(\mathcal{A})$ is injectively cofibration.

The similar things hold for the second statement.

2.3 Homotopy limits for simplicial functors

For the theory of ∞ -categories, we refer to [20]. Let \mathbf{A} be a combinatorial simplicial model category and \mathcal{I} be a small simplicial category. We denote by $\mathbf{A}^{\mathcal{I}}$ the category of simplicial functors $F : \mathcal{I} \rightarrow \mathbf{A}$. Then by [20, Proposition A.3.2.2] the category $\mathbf{A}^{\mathcal{I}}$ admits two structures of Quillen model category whose weak equivalences are the term-wise weak equivalences:

1. The projective model structure, whose fibrations are the term-wise fibrations.
2. The injective model structure, whose cofibrations are the term-wise cofibrations.

For an object $M \in \mathbf{A}$, we denote by $\text{ct}_{\mathcal{I}} M$ the constant functor $\mathcal{I} \rightarrow \mathbf{A}$ with the value M . If the diagram category \mathcal{I} is clear from the context, we often omit the subscript \mathcal{I} and denote by $\text{ct } M$.

Let $\text{ct} : \mathbf{A} \rightarrow \mathbf{A}^{\mathcal{I}}$ be the functor which sends M to $\text{ct } M$. Then by [20, Proposition A. 3.3.8] we have Quillen adjunction

$$\text{ct} : \mathbf{A} \rightleftarrows \mathbf{A}^{\mathcal{I}} : \lim$$

where we equip $\mathbf{A}^{\mathcal{I}}$ with the injective model structure.

Let F be an object of $\mathbf{A}^{\mathcal{I}}$ and X be an object of \mathbf{A} . We say that a morphism $\eta : \text{ct } X \rightarrow F$ in $\mathbf{A}^{\mathcal{I}}$ *exhibits X as a homotopy limit of F* if, for some equivalence $F \xrightarrow{\sim} F'$ where F' is injective fibrant in $\mathbf{A}^{\mathcal{I}}$, the composite map $X \rightarrow \lim F \rightarrow \lim F'$ is weak equivalence.

Since \lim preserves a weak equivalence between injectively fibrant objects, this condition is independent to the choice of F' . We can easily prove the following lemma.

Lemma 2.15. (1) *If F is injective fibrant object of \mathbf{A} , then the canonical map $\text{ct } \lim F \rightarrow F$ exhibits $\lim F$ as a homotopy limit of F .*

(2) *Let $\psi : X' \rightarrow X$ be a weak equivalence of \mathbf{A} . Then the morphism $\eta : \text{ct } X \rightarrow F$ exhibits X as a homotopy limit of F if and only if the composite morphism $\eta \circ \text{ct}(\psi) : \text{ct } X' \rightarrow \text{ct } X \rightarrow F$ exhibits X' as a homotopy limit of F .*

(3) *Let $F \xrightarrow{\sim} G$ be a weak equivalence in $\mathbf{A}^{\mathcal{I}}$. Assume that a morphism $\eta : \text{ct } X \rightarrow F$ is given for some X . Then the morphism η exhibits X as a homotopy limit of F if and only if the composite morphism $\text{ct } X \rightarrow F \rightarrow G$ exhibits X as a homotopy limit of G .*

When we study limits in category theory, two important notions are cofinality of functors and filtered categories. For the definition of filtered ∞ -category, we refer to [20, Definition 5.3.1.7]. We call an ∞ -category \mathcal{C} *co-filtered* if the opposite ∞ -category \mathcal{C}^{op} is filtered. A ∞ -category version of cofinality was introduced by Joyal [20, Definition 4.1.1.1]. In the present paper we mainly use the dual notion left cofinality. An ∞ -functor $F : A \rightarrow B$ is called *left cofinal* if the opposite functor $F^{\text{op}} : A^{\text{op}} \rightarrow B^{\text{op}}$ is a cofinal ∞ -functor. A simplicial functor $\phi : \mathcal{I} \rightarrow \mathcal{J}$ between fibrant simplicial categories is called *homotopy left cofinal* if the simplicial nerve $N(\phi) : N(\mathcal{I}) \rightarrow N(\mathcal{J})$ is left cofinal. We note that Dwyer-Kan equivalence of fibrant simplicial categories is an important example of homotopy left cofinal functor.

For simplicial functors $\phi : \mathcal{I} \rightarrow \mathcal{J}$ and $F : \mathcal{J} \rightarrow \mathbf{A}$, we often denote by $F|_{\mathcal{I}}$ the composite functor $F \circ \phi : \mathcal{I} \rightarrow \mathbf{A}$.

Lemma 2.16. *Let $\mathcal{I} \rightarrow \mathcal{J}$ be a homotopy left cofinal functor between fibrant simplicial categories. Assume that every object of \mathbf{A} is cofibrant. Then the morphism $\eta : \text{ct}_{\mathcal{J}} X \rightarrow F$ exhibits X as a homotopy limit of F if and only if $\eta|_{\mathcal{I}} : \text{ct}_{\mathcal{I}} X \rightarrow F|_{\mathcal{I}}$ exhibits X as a homotopy limit of $F|_{\mathcal{I}}$.*

Proof. First note that in the case when X and the every value of F are fibrant-cofibrant in \mathbf{A} this lemma is proved by [20, Proposition 4.1.1.8, Theorem 4.2.4.2]. We also note that by the same consideration of [20, Remark A.2.6.8] we see that each value $F(i)$ of an injectively fibrant object F of $\mathbf{A}^{\mathcal{J}}$ is a fibrant object of \mathbf{A} . Hence it follows from the assumption that each value of injectively fibrant object F is a fibrant-cofibrant object of \mathbf{A} .

We assume that η exhibits X as a homotopy limit of F . We take an injective trivial cofibration $F \xrightarrow{\sim} F'$ with F' injectively fibrant. The map $X \rightarrow \lim F'$ induced from the composite map $\text{ct } X \rightarrow F \rightarrow F'$ a weak equivalence. Let $F|_{\mathcal{I}} \xrightarrow{\sim} G$ be a weak equivalence with G injectively fibrant. Since

the restriction functor $\mathbf{A}^{\mathcal{J}} \rightarrow \mathbf{A}^{\mathcal{I}}$ preserves injective trivial cofibrations, by the lifting property we obtain the commutative diagram

$$\begin{array}{ccccc} \mathrm{ct}_{\mathcal{I}} X & \longrightarrow & F|_{\mathcal{I}} & \xrightarrow{\sim} & G \\ \downarrow \wr & & \downarrow \wr & \nearrow \sim & \downarrow \\ \mathrm{ct}_{\mathcal{I}}(\lim_{\mathcal{J}} F') & \longrightarrow & F'|_{\mathcal{I}} & \longrightarrow & \emptyset \end{array}$$

Since $\mathrm{ct}_{\mathcal{J}}(\lim_{\mathcal{J}} F') \rightarrow F'$ exhibits $\lim F'$ as a homotopy limit of F' , by the known results mentioned above the map $\mathrm{ct}_{\mathcal{I}}(\lim_{\mathcal{J}} F') \rightarrow F'|_{\mathcal{I}}$ exhibits $\lim F'$ as a homotopy limit. Hence the map $\lim_{\mathcal{J}} F' \rightarrow \lim_{\mathcal{I}} G$ induced from the composite map $\mathrm{ct}_{\mathcal{I}} \lim F' \rightarrow F'|_{\mathcal{I}} \rightarrow G$ is weak equivalence. Consequently we see that the map $\mathrm{ct}_{\mathcal{I}} X \rightarrow \lim_{\mathcal{I}} G$ is weak equivalence.

We assume that $\eta|_{\mathcal{I}}$ exhibits X as a homotopy limit of $F|_{\mathcal{I}}$. It is easy to see that we have only to check the case when a weak equivalence $F \rightarrow F'$ is an injective trivial cofibration. Take a weak equivalence $F|_{\mathcal{I}} \xrightarrow{\sim} G$ with G injectively fibrant in $\mathbf{A}^{\mathcal{I}}$. Then as before by the lifting property we obtain the commutative diagram

$$\begin{array}{ccccc} \mathrm{ct}_{\mathcal{I}} X & \longrightarrow & F|_{\mathcal{I}} & \xrightarrow{\sim} & G \\ \downarrow & & \downarrow \wr & \nearrow \sim & \downarrow \\ \mathrm{ct}_{\mathcal{I}}(\lim_{\mathcal{J}} F') & \longrightarrow & F'|_{\mathcal{I}} & \longrightarrow & \emptyset \end{array}$$

Since F' is taken to be injectively fibrant, the canonical map $\mathrm{ct}_{\mathcal{J}}(\lim_{\mathcal{J}} F') \rightarrow F'$ exhibits $\lim F'$ as a homotopy limit of F' . Therefore by the known result mentioned above, the canonical map $\mathrm{ct}_{\mathcal{I}}(\lim_{\mathcal{J}} F') \rightarrow F'|_{\mathcal{I}}$ exhibits $\lim_{\mathcal{J}} F'$ as a homotopy limit of $F'|_{\mathcal{I}}$. Hence the map $\lim_{\mathcal{J}} F' \rightarrow \lim_{\mathcal{I}} G$ induced from the above diagram is weak equivalence. By the assumption the map $X \rightarrow \lim_{\mathcal{I}} G$ is weak equivalence. Consequently, the map $X \rightarrow \lim_{\mathcal{J}} F'$ is weak equivalence as desired. \square

Let $f : \mathbf{A} \rightleftarrows \mathbf{B} : g$ be a Quillen adjunction between combinatorial simplicial model categories. We define $f^{\mathcal{I}} : \mathbf{A}^{\mathcal{I}} \rightarrow \mathbf{B}^{\mathcal{I}}$ to be the functor which sends $G \in \mathbf{A}$ to $f \circ G$ and define $g^{\mathcal{I}} : \mathbf{B}^{\mathcal{I}} \rightarrow \mathbf{A}^{\mathcal{I}}$ similarly. Then $f^{\mathcal{I}} : \mathbf{A}^{\mathcal{I}} \rightleftarrows \mathbf{B}^{\mathcal{I}} : g^{\mathcal{I}}$ is a Quillen adjunction where we equip \mathbf{A} and \mathbf{B} with the injective model structures.

Lemma 2.17. *Assume that the simplicial functor g preserves weak equivalence. If a morphism $\eta : \mathrm{ct}_{\mathcal{I}} X \rightarrow F$ in $\mathbf{A}^{\mathcal{I}}$ exhibits X as a homotopy limit of F , Then $g^{\mathcal{I}}(\eta) : \mathrm{ct}_{\mathcal{I}} g(X) \rightarrow g^{\mathcal{I}}(F)$ exhibits $g(X)$ as a homotopy limit of $g^{\mathcal{I}}F$.*

Proof. We take an injectively trivial cofibration $g^{\mathcal{I}}(F) \xrightarrow{\sim} G$ with G injectively fibrant. Let $F \xrightarrow{\sim} F'$ be a weak equivalence in $\mathbf{A}^{\mathcal{I}}$ with F' injectively fibrant. Since $g^{\mathcal{I}}(F')$ is injectively fibrant, by the lifting property we obtain the commutative diagram

$$(2-5) \quad \begin{array}{ccccc} \mathrm{ct}_{\mathcal{I}} g(X) & \longrightarrow & g^{\mathcal{I}}(F) & \xrightarrow{\sim} & g^{\mathcal{I}}(F') \\ & \searrow & \downarrow \wr & \nearrow & \\ & & G & & \end{array}$$

By the assumption on g , the induced morphism $g^{\mathcal{I}}(F) \rightarrow g^{\mathcal{I}}(F')$ is weak equivalence. Therefore the morphism $G \rightarrow g^{\mathcal{I}}(F')$ in the above diagram is also weak equivalence. Since \lim is a right Quillen functor, it preserves a weak equivalence between fibrant object. Thus taking limits of the above diagram (2-5) we see that the induced morphism $g(X) \rightarrow G$ is weak equivalence. \square

In the case when a diagram category \mathcal{I} is of special type, we can easily compute homotopy limits of functors $F : \mathcal{I} \rightarrow \mathbf{A}$. We set $\mathcal{I} := (\mathbb{Z}_{\geq 1})^{\text{op}}$ where we consider the totally ordered set $\mathbb{Z}_{\geq 1}$ as a category in the standard way. Namely, the objects of \mathcal{I} are positive integers and morphisms $\mathbf{geq}^{m,n} : m \rightarrow n$ are $m \geq n$. Using [8, Theorem 15.10.12] we prove the following lemma. Here we equip $\mathcal{C}(\mathcal{A})$ with the projective model structure.

Lemma 2.18. *Let \mathcal{A} be a small dg category. Let F be an object of $\mathcal{C}_{\text{ss}}(\mathcal{A})^{\mathcal{I}}$. Assume that for $n \geq 1$ the morphism $F(\mathbf{geq}^{n+1,n}) : F(n+1) \rightarrow F(n)$ is a projective fibration. Then the canonical map $\text{ct lim } F \rightarrow F$ exhibits $\text{lim } F$ as a homotopy limit.*

We often use the following characterization of homotopy left cofinality.

Theorem 2.19 (Theorem 4.1.3.1 [20]). *Let $f : \mathcal{C} \rightarrow \mathcal{D}$ be a map of simplicial sets, where \mathcal{D} is an ∞ -category. The following conditions are equivalent:*

- (1) *f is left cofinal.*
- (2) *For every $D \in \mathcal{D}$, the simplicial set $f_{/D} := \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/D}$ is weakly contractible.*

Lemma 2.20. *Let \mathcal{C} be a co-filtered ∞ -category and $\mathcal{D} \subset \mathcal{C}$ a full sub ∞ -category. Assume that for each object c of \mathcal{C} there exists a morphism $d \rightarrow c$ in \mathcal{C} the domain of which is an object of \mathcal{D} . Then \mathcal{D} is left cofinal in \mathcal{C} .*

Proof. By the above Theorem 2.19 it is enough to prove that the simplicial set $\mathcal{D}_{/c}$ is weakly contractible for each $c \in \mathcal{C}$. By the assumption $\mathcal{D}_{/c}$ is non-empty. Therefore we have only to show that any map $\sigma : K \rightarrow \mathcal{D}_{/c}$ of simplicial sets from a finite simplicial set K factors through the canonical map $\iota : K \rightarrow {}^{\triangleleft}K$.

$$\begin{array}{ccc} K & \xrightarrow{\sigma} & \mathcal{D}_{/c} \\ \downarrow \iota & \nearrow & \\ {}^{\triangleleft}K & & \end{array}$$

Recall that $\mathcal{D}_{/c} = \mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{/c}$. We denote by σ_1, σ_2 the composite morphism $\text{pr}_1 \circ \sigma, \text{pr}_2 \circ \sigma$ where pr_1, pr_2 are the first and the second projections.

$$\begin{array}{ccccc} K & & & & \\ \sigma_1 \searrow & \sigma \searrow & \sigma_2 \searrow & & \\ & \mathcal{D}_{/c} & \xrightarrow{\text{pr}_2} & \mathcal{C}_{/c} & \\ \text{pr}_1 \downarrow & & & \downarrow \Upsilon & \\ & \mathcal{D} & \longrightarrow & \mathcal{C} & \end{array}$$

Since \mathcal{C} is co-filtered, so is $\mathcal{C}_{/c}$. Therefore σ_2 factors through ι . In other word there exists a map $\psi : {}^{\triangleleft}K \rightarrow \mathcal{C}_{/c}$ of simplicial sets such that $\sigma_2 = \psi \circ \iota$. We denote by $f : c' \rightarrow c$ the image of the cone point $-\infty$ of ${}^{\triangleleft}K$ by ψ . We may consider f as a morphism of \mathcal{C} the co-domain of which is c . We denote by c' the domain of the morphism f . By assumption there exists a morphism $g : d \rightarrow c'$ in \mathcal{C} the domain of which belongs to \mathcal{D} . We may consider g gives a morphism in $\mathcal{C}_{/c}$ from some object $h : d \rightarrow c$ to $c' \rightarrow c$. By abusing notation we denote by g the corresponding map $g : \Delta^1 \rightarrow \mathcal{C}_{/c}$ of simplicial sets.

Since the restriction $g|_{\{1\}}$ to $\{1\} \subset \Delta^1$ coincide with the restriction $\sigma|_{\{-\infty\}}$ to $\{-\infty\} \subset {}^{\triangleleft}K$, the maps σ_2 and g induce the map $\Sigma : \Delta^1 \sqcup_{\{1\}} {}^{\triangleleft}K \rightarrow \mathcal{C}_{/c}$ of simplicial sets. By [20, Lemma 2.4.3.1] the inclusion $\eta : \Delta^1 \sqcup_{\{1\}} {}^{\triangleleft}K \rightarrow \Delta^1 \star K$ is inner anodyne. Therefore there exists a map

$\Sigma' : \Delta^1 \star K \rightarrow \mathcal{C}$ such that $\Sigma = \Sigma' \circ \eta$. We denote by τ_2 the composite map of Σ'' with the canonical map ${}^\triangleleft K \cong \{0\} \star K \rightarrow \Delta^1 \star K$.

Since the restriction $\tau_2|_K$ to K of τ coincide with σ_2 and the subcategory \mathcal{D} of \mathcal{C} is full, the image of composite map $\Upsilon \circ \tau_2$ of τ_2 with the domain functor $\Upsilon : \mathcal{C}_{/c} \rightarrow \mathcal{C}$ contained in \mathcal{D} . Therefore the map $\Upsilon \circ \tau_2$ is decomposed into some map $\tau_1 : {}^\triangleleft K \rightarrow \mathcal{D}$ followed by the inclusion $\mathcal{D} \subset \mathcal{C}$. Since the restriction $\tau_2|_K$ to K of τ coincides with σ_2 , the restriction $\tau_1|_K$ to K coincide with σ_1 . Therefore the maps τ_1, τ_2 gives a map $\tau : {}^\triangleleft K \rightarrow \mathcal{D}_{/c}$ such that $\tau \circ \iota = \sigma$. \square

3 Derived bi-duality via homotopy limit

3.1 Derived bi-commutator categories and dualities

Let \mathcal{A} be a small locally cofibrant dg-category and $\mathcal{J} \subset \mathcal{D}(\mathcal{A})$ a small full subcategory. Following Efimov [5] we construct the derived bi-commutator category $\mathcal{Bic}_{\mathcal{A}}(\mathcal{J})$ in the following steps.

Construction 3.1. (I). First we choose an injectively fibrant representative J of each object of \mathcal{J} , and denote by $\underline{\mathcal{J}}_1$ the full subcategory of $\mathcal{C}(\mathcal{A})$ consisting of those representatives. Then we define the opposite of endomorphism dg-category $\mathcal{E} := \mathcal{E}nd_{\mathcal{A}}(\underline{\mathcal{J}}_1)^{op}$ to be

$$Ob(\mathcal{E}) := Ob(\underline{\mathcal{J}}_1), \quad Hom_{\mathcal{E}}([J], [I]) := {}_{\mathcal{A}}(I, J).$$

where $[I]$ and $[J]$ are objects of \mathcal{E} which correspond to $I, J \in Ob(\underline{\mathcal{J}}_1)$ respectively. Note that by [5, Lemma 3.1] the quasi-equivalence class of \mathcal{E} is only depends on \mathcal{J} and is independent to the choice of the representative $\underline{\mathcal{J}}_1$. Note also that \mathcal{E} is a locally cofibrant dg-category.

(II). Next we define a dg $\mathcal{A} \otimes \mathcal{E}$ -module $\underline{\mathcal{J}}_2$ to be $\underline{\mathcal{J}}_2(a \otimes [J]) := J(a)$ for $a \in \mathcal{A}$ and $[J] \in \mathcal{E}$. We call $\underline{\mathcal{J}}_2$ the *diagonal module* associated to $\underline{\mathcal{J}}_1$. Note that the diagonal module $\underline{\mathcal{J}}_2$ is determined by $\underline{\mathcal{J}}_1$.

(III). Finally we take a fibrant replacement $\underline{\mathcal{J}}_2 \xrightarrow{\sim} \underline{\mathcal{J}}$ of $\underline{\mathcal{J}}_2$ in $\mathcal{C}(\mathcal{A} \otimes \mathcal{E})$.

Definition 3.2. We define the bi-commutator category $\mathcal{Bic}_{\mathcal{A}}(\mathcal{J})$ of \mathcal{J} over \mathcal{A} to be

$$Ob(\mathcal{Bic}_{\mathcal{A}}(\mathcal{J})) := Ob\mathcal{A}, \quad \mathcal{Bic}_{\mathcal{A}}(\mathcal{J})(a, b) := {}_{\mathcal{E}}(\underline{\mathcal{J}}(b \otimes -), \underline{\mathcal{J}}(a \otimes -)).$$

The identity on the objects induces a canonical functor $\iota : \mathcal{A} \rightarrow \mathcal{Bic}_{\mathcal{A}}(\mathcal{J})$. We denote by $\mathbb{Bic}_{\mathcal{A}}(\mathcal{J})$ the object of $Ho(dgCat)$ corresponding to $\mathcal{Bic}_{\mathcal{A}}(\mathcal{J})$. Note that by [5, Lemma 3.2] the object $\iota : \mathcal{A} \rightarrow \mathcal{Bic}_{\mathcal{A}}(\mathcal{J})$ of $Ho(dgCat_{\mathcal{A}'})$ only depends on the subcategory $\mathcal{J} \subset \mathcal{D}(\mathcal{A})$.

We introduce the duality functors induced by the $\mathcal{A} \otimes \mathcal{E}$ -module $\underline{\mathcal{J}}$:

$$\begin{aligned} D &:= {}_{\mathcal{A}}(-, \underline{\mathcal{J}}) : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{E})^{op} \\ D' &:= {}_{\mathcal{E}}(-, \underline{\mathcal{J}}) : \mathcal{C}(\mathcal{E}) \rightarrow \mathcal{C}(\mathcal{A})^{op} \end{aligned}$$

We set $S := D'D$. Note that $S(M)$ computes the derived bi-duality of M over \mathcal{J} (Corollary 3.7). Namely the module $S(M)$ is isomorphic to $\mathbb{R} Hom_{\mathcal{E}}(\mathbb{R} Hom_{\mathcal{A}}(M, \mathcal{J}), \mathcal{J})$ in the derived category $\mathcal{D}(\mathcal{A})$.

There exist natural transformations $\epsilon : 1_{\mathcal{C}(\mathcal{A})} \rightarrow S$ induced from the evaluation morphisms. Namely, for $a \in \mathcal{A}$ and $M \in \mathcal{C}(\mathcal{A})$, the morphism

$$\epsilon_M(a) : M(a) \rightarrow S(M)(a) = {}_{\mathcal{E}}({}_{\mathcal{A}}(M, \underline{\mathcal{J}}), \underline{\mathcal{J}}(a \otimes -))$$

of cochain complexes is given for a homogeneous elements $m \in M(a)$ by

$$[\epsilon_M(a)(m)](e) : {}_{\mathcal{A}}(M, \underline{\mathcal{J}}(- \otimes e)) \rightarrow \underline{\mathcal{J}}(a \otimes e), \quad f \mapsto (-1)^{(\deg m)(\deg f)} f(m)$$

where $e \in \mathcal{E}$ and $f \in \mathcal{A}(M, J(- \otimes e))$ is a homogeneous element.

We can check that the family of morphisms

$$\mathcal{A}(a, b) = b^\wedge(a) \xrightarrow{\epsilon_{b^\wedge}(a)} S(b^\wedge)(a) = \varepsilon(\underline{\mathcal{J}}(b \otimes -), \underline{\mathcal{J}}(a \otimes -)) = \mathcal{Bic}_{\mathcal{A}}(J)(a, b)$$

gives a dg-functor $\iota : \mathcal{A} \rightarrow \mathcal{Bic}_{\mathcal{A}}(\mathcal{J})$. We immediately see the following lemma.

Lemma 3.3. *The functor ι is quasi-equivalence if and only if ϵ_{b^\wedge} is quasi-isomorphism for $b \in \mathcal{A}$.*

To state the main theorem of this paper, we prepare notations. We denote by $\langle \mathcal{J} \rangle^{\mathcal{D}}$ the smallest thick subcategory of $\mathcal{D}(\mathcal{A})$ containing \mathcal{J} . Namely it is the full subcategory of $\mathcal{D}(\mathcal{A})$ which consists of those objects which are obtained by taking direct summands, shifts and cones finitely many times from the image of \mathcal{J} by the canonical functor $\mathcal{C}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{A})$. We denote by $\langle \mathcal{J} \rangle^{\mathcal{C}}$ the full subcategory of $\mathcal{C}(\mathcal{A})$ consisting those objects which are sent to $\langle \mathcal{J} \rangle^{\mathcal{D}}$ by the canonical functor $\mathcal{C}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{A})$.

We fix a dg \mathcal{A} -module M . We define a full subcategory $\mathcal{U} := \mathcal{U}_{\mathcal{J}}^M$ of $\langle \mathcal{J} \rangle_{M/}^{\mathcal{C}}$ to be the full subcategory $(\langle \mathcal{J} \rangle_{M/}^{\mathcal{C}})^{\circ}$ consisting of fibrant-cofibrant objects. Namely \mathcal{U} is the full subcategory of $\langle \mathcal{J} \rangle_{M/}^{\mathcal{C}}$ consisting those object $k : M \rightarrow K$ such that the morphism k is a injective cofibration and the co-domain K is injective fibrant. Let $\Gamma : \mathcal{U} \rightarrow \mathcal{C}(\mathcal{A})$ be the co-domain functor.

$$\Gamma : \mathcal{U} \rightarrow \mathcal{C}(\mathcal{A}), \quad [k : M \rightarrow K] \mapsto K.$$

We observe that the assignment $\kappa : k \mapsto k$ gives a natural transformation $\kappa : \text{ct}_{\mathcal{U}} M \rightarrow \Gamma$ where $\text{ct}_{\mathcal{U}} M : \mathcal{U} \rightarrow \mathcal{C}(\mathcal{A}), k \mapsto M$ is the constant functor with the value M .

$$\begin{array}{ccc} (\text{ct}_{\mathcal{U}} M)(k) & \xlongequal{\quad} & M \\ \kappa_k \downarrow & & \downarrow k \\ \Gamma(k) & \xlongequal{\quad} & K \end{array}$$

We have the commutative diagram

$$\begin{array}{ccc} \text{ct}_{\mathcal{U}} M & \xrightarrow{\epsilon_M} & \text{ct}_{\mathcal{U}} S(M) \\ \kappa \downarrow & & \downarrow S(\kappa) \\ \Gamma & \xrightarrow[\epsilon_{\Gamma}]{\sim} & S \circ \Gamma \end{array}$$

Then the main theorem is the following.

Theorem 3.4. *Let \mathcal{A} be a locally cofibrant dg-category, M a dg \mathcal{A} -module and \mathcal{J} a set of objects of $\mathcal{D}(\mathcal{A})$. We construct a derived bi-duality functor $S(-)$ over \mathcal{J} by the construction 3.1. Assume that the evaluation map $\epsilon_M : M \rightarrow S(M)$ is injectively cofibrant. Then we have a quasi-isomorphism*

$$S(M) \simeq \text{holim}_{\mathcal{U}} \Gamma$$

More precisely, the morphism $S(\kappa) : \text{ct}_{\mathcal{U}} S(M) \rightarrow S \circ \Gamma$ exhibits $S(M)$ as a homotopy limit of $S \circ \Gamma$. Since the morphism $\epsilon_{\Gamma} : \Gamma \rightarrow S \circ \Gamma$ is a weak equivalence (Lemma 3.8.(1)), we obtain the above quasi-isomorphism.

The assumption that ϵ_M is injective cofibration is harmless, because of the following lemma.

Lemma 3.5. *Let \mathcal{A} and \mathcal{J} as in the above theorem. We fix a set \mathcal{M} of objects of $\mathcal{C}(\mathcal{A})$. Then by the construction 3.1 we can construct a derived bi-duality functor $S(-)$ over \mathcal{J} such that the evaluation map ϵ_M is injective cofibration for $M \in \mathcal{M}$.*

Proof. We claim that there exists a injective cofibration $M \hookrightarrow N_M$ with N_M injectively fibrant and weakly contractible. Indeed first we take a fibrant replacement $M \xrightarrow{\sim} M'$ of M by a injectively cofibration. Let $M' \rightarrow c(1_{M'})$ be the canonical embedding. Then by Lemma 2.6 the composite map $M \rightarrow M' \rightarrow c(1_{M'})$ satisfies the desired property.

In the step (I) first we choose injectively fibrant representatives of \mathcal{J} and construct $\underline{\mathcal{J}}'_1$. Next for some $J'_0 \in \underline{\mathcal{J}}'_1$ we set $J_0 := J'_0 \oplus \bigoplus_{M \in \mathcal{M}} N_M$. Then we obtain $\underline{\mathcal{J}}_1$ such that for each $M \in \mathcal{M}$, there exists an injective cofibration $\iota : M \hookrightarrow J_0$ for some object J_0 of $\underline{\mathcal{J}}_1$. Let $\underline{\mathcal{J}}_2$ is the diagonal module of $\underline{\mathcal{J}}_0$ and $\underline{\mathcal{J}}$ an injectively fibrant replacement by an injective trivial cofibration $\underline{\mathcal{J}}_2 \xrightarrow{\sim} \underline{\mathcal{J}}$. Then for any $M \in \mathcal{M}$ there exists an injective cofibration $M \hookrightarrow \underline{\mathcal{J}}(- \otimes [J_0])$. Now it is easy to see by the explicit formula of the evaluation map that $\epsilon_M : M \rightarrow S(M)$ is an injectively cofibration. \square

3.2 A proof of Theorem 3.4

First we study basic properties of duality D, D' .

Lemma 3.6. (1) For each $e \in \mathcal{E}$ the dg \mathcal{A} -module $\underline{\mathcal{J}}(- \otimes e)$ is injectively fibrant. Dually for each $a \in \mathcal{A}$ the dg \mathcal{E} -module $\underline{\mathcal{J}}(a \otimes -)$ is injectively fibrant.

(2) The functors D, D' preserves quasi-isomorphisms.

(3) The functor S preserves quasi-isomorphisms, projectively fibrations and injective cofibrations.

(4) If we equip $\mathcal{C}(\mathcal{A})$ with the injective model structure and $\mathcal{C}(\mathcal{E})^{\text{op}}$ with the opposite model structure of the projective model structure in $\mathcal{C}(\mathcal{E})$, then the adjoint pair

$$D : \mathcal{C}(\mathcal{A}) \rightleftarrows \mathcal{C}(\mathcal{E})^{\text{op}} : D'$$

is a Quillen adjunction.

Proof. (1) follows from Lemma 2.11.

(2) and (3) follow from (1) and Lemma 2.4.

(4) By (2) it is enough to prove that D sends injective cofibration to projective fibration. This follows (1) and Lemma 2.4. \square

Corollary 3.7. For any dg \mathcal{A} -module M , the module $S(M)$ computes the derived bi-duality over \mathcal{J} .

$$S(M) \simeq \mathbb{R} \text{Hom}_{\mathcal{E}}(\mathbb{R} \text{Hom}_{\mathcal{A}}(M, \mathcal{J}), \mathcal{J}).$$

We denote by $\text{Perf}_{\text{dg}} \mathcal{E}$ the category $\langle e^\wedge \mid e \in \mathcal{E} \rangle^c$ of perfect dg \mathcal{E} -modules. There exists a natural transformation $\epsilon' : 1_{\mathcal{C}(\mathcal{E})} \rightarrow DD'$ induced from evaluation maps.

Lemma 3.8. (1) If a dg \mathcal{A} -module K belongs to $\langle \mathcal{J} \rangle^c$, then the morphism $\epsilon_K : K \rightarrow S(K)$ is a quasi-isomorphism.

(2) If a dg- \mathcal{E} -module P belongs to $\text{Perf}_{\text{dg}} \mathcal{E}$, then the morphism $\epsilon'_P : P \rightarrow DD'(P)$ is a quasi-isomorphism.

(3) The functors D, D' gives a contravariant Dwyer-Kan equivalence between $(\langle \mathcal{J} \rangle^c)^\circ$ and $(\text{Perf}_{\text{dg}} \mathcal{E})^\circ$.

Proof. (1) It is enough to prove the case when $K = J$ for some $J \in \underline{\mathcal{J}}_1$. It is easy to see that the dg \mathcal{E} -module ${}_{\mathcal{A}}(J, \underline{\mathcal{J}}_2)$ is isomorphic to $[J]^\wedge$ and that the evaluation morphism

$$\varepsilon_J : J \rightarrow \varepsilon({}_{\mathcal{A}}(J, \underline{\mathcal{J}}_2), \underline{\mathcal{J}}_2)$$

is an isomorphism of dg \mathcal{A} -modules. Let $\phi : \underline{\mathcal{J}}_2 \xrightarrow{\sim} \underline{\mathcal{J}}$ be the replacement morphism. We can check that the morphism $\psi_1 : \varepsilon({}_{\mathcal{A}}(J, \underline{\mathcal{J}}_2), \underline{\mathcal{J}}_2) \rightarrow \varepsilon({}_{\mathcal{A}}(J, \underline{\mathcal{J}}_2), \underline{\mathcal{J}})$ of dg \mathcal{A} -modules induced from ϕ is isomorphic to the morphism $\phi(- \otimes J) : \underline{\mathcal{J}}_2(- \otimes [J]) \rightarrow \underline{\mathcal{J}}(- \otimes [J])$ under the isomorphisms

$$\varepsilon({}_{\mathcal{A}}(J, \underline{\mathcal{J}}_2), \underline{\mathcal{J}}_2) \cong \underline{\mathcal{J}}_2(- \otimes [J]), \quad \varepsilon({}_{\mathcal{A}}(J, \underline{\mathcal{J}}_2), \underline{\mathcal{J}}) \cong \underline{\mathcal{J}}(- \otimes [J]).$$

This implies that ψ_1 is a quasi-isomorphism. Let J be an object of $\underline{\mathcal{J}}_1$. By the definition of the diagonal module $\underline{\mathcal{J}}_2$, the \mathcal{A} -module $\underline{\mathcal{J}}_2(- \otimes [J])$ is isomorphic to the \mathcal{A} -module. Recall that all objects of $\underline{\mathcal{J}}_1 \subset \mathcal{C}(\mathcal{A})$ are taken to be injectively fibrant. Hence the \mathcal{A} -module $\underline{\mathcal{J}}_2(- \otimes e)$ is injectively fibrant for each $e \in \text{Ob}(\mathcal{E})$. By Lemma 3.6, the \mathcal{A} -module $\underline{\mathcal{J}}(- \otimes e)$ is injectively fibrant for each $e \in \text{Ob}(\mathcal{E})$. Therefore by Lemma 2.12 the induced map ${}_{\mathcal{A}}(J, \underline{\mathcal{J}}_2) \rightarrow {}_{\mathcal{A}}(J, \underline{\mathcal{J}})$ is quasi-isomorphism of \mathcal{E} -modules. By Lemma 3.6, we see that the induced morphism $\psi_2 : \varepsilon({}_{\mathcal{A}}(J, \underline{\mathcal{J}}), \underline{\mathcal{J}}) \rightarrow \varepsilon({}_{\mathcal{A}}(J, \underline{\mathcal{J}}_2), \underline{\mathcal{J}})$ is a quasi-isomorphism.

We can check that $\psi_2 \circ \epsilon_J = \psi_1 \circ \varepsilon_J$.

$$(3-6) \quad \begin{array}{ccc} J & \xrightarrow{\epsilon_J} & \varepsilon({}_{\mathcal{A}}(J, \underline{\mathcal{J}}), \underline{\mathcal{J}}) \\ \varepsilon_J \downarrow \sim & & \sim \downarrow \psi_2 \\ \varepsilon({}_{\mathcal{A}}(J, \underline{\mathcal{J}}_2), \underline{\mathcal{J}}_2) & \xrightarrow[\psi_1]{\sim} & \varepsilon({}_{\mathcal{A}}(J, \underline{\mathcal{J}}_2), \underline{\mathcal{J}}) \end{array}$$

Since the morphisms $\varepsilon_J, \psi_1, \psi_2$ are quasi-isomorphisms, we conclude that ϵ_J is a quasi-isomorphism.

(2) It is enough to check the case $P = [J]^\wedge$ for some $J \in \underline{\mathcal{J}}_1$. In the similar way as in (1), we obtain the commutative diagram similar to the above diagram (3-6)

$$\begin{array}{ccc} [J]^\wedge & \xrightarrow{\epsilon'_{[J]^\wedge}} & {}_{\mathcal{A}}(\varepsilon([J]^\wedge, \underline{\mathcal{J}}), \underline{\mathcal{J}}) \\ \epsilon'_{[J]^\wedge} \downarrow \sim & & \sim \downarrow \psi'_2 \\ {}_{\mathcal{A}}(\varepsilon([J]^\wedge, \underline{\mathcal{J}}_2), \underline{\mathcal{J}}_2) & \xrightarrow[\psi'_1]{\sim} & {}_{\mathcal{A}}(\varepsilon([J]^\wedge, \underline{\mathcal{J}}_2), \underline{\mathcal{J}}) \end{array}$$

By Yoneda Lemma we have an isomorphism of \mathcal{A} -modules $\varepsilon([J]^\wedge, \underline{\mathcal{J}}_2) \cong \underline{\mathcal{J}}_2(- \otimes [J]) \cong J$. Observe that the left vertical arrow $\epsilon'_{[J]^\wedge}$ coincide with the isomorphism

$$[J]^\wedge \cong {}_{\mathcal{A}}(J, \underline{\mathcal{J}}_2) \cong {}_{\mathcal{A}}(\varepsilon([J]^\wedge, \underline{\mathcal{J}}_2), \underline{\mathcal{J}}_2).$$

Hence it is an isomorphism. Since $[J]^\wedge$ is projectively cofibrant, the induced map $\varepsilon([J]^\wedge, \underline{\mathcal{J}}_2) \rightarrow \varepsilon([J]^\wedge, \underline{\mathcal{J}})$ is a quasi-isomorphism by Lemma 2.12. Hence by Lemma 3.6 the left vertical morphism ψ'_2 is a quasi-isomorphism. We can prove that the bottom arrow ψ'_1 is a quasi-isomorphism in the same way of the proof that ψ_2 is quasi-isomorphism in (1). Thu we conclude that the evaluation map $\epsilon'_{[J]^\wedge}$ is quasi-isomorphism. Now we complete the proof of (2).

(3) follows from (1) and (2). \square

Now we start to prove Theorem 3.4. We equip $\mathcal{C}(\mathcal{E})$ with the projective model structure. We define the full sub category \mathcal{O} to be $(\text{Perf}_{\text{dg}} \mathcal{E}_{/DM})^\circ$. Namely \mathcal{O} is the full subcategory of $\text{Perf}_{\text{dg}} \mathcal{E}_{/DM}$ consisting of those object $p : P \rightarrow DM$ such that the morphism p is a projective fibration and the domain P is projective cofibrant in $\mathcal{C}(\mathcal{E})$.

We construct a simplicial functor $\widetilde{D}' : \mathcal{O}^{\text{op}} \rightarrow \mathcal{U}$. The pair D, D' of dg-functors is a contravariant adjunction. Namely there exists a natural isomorphism of Hom complexes

$$\widetilde{D}'_{N,M} : \varepsilon(N, DM) \xrightarrow{\cong} {}_{\mathcal{A}}(M, D'N)$$

for $M \in \mathcal{C}(\mathcal{A})$ and $N \in \mathcal{C}(\mathcal{E})$.

For an object $p : P \rightarrow DM$ of \mathcal{O} , we define $\widetilde{D}'(p)$ to be $\widetilde{D}'(p) := \widetilde{D}'_{P,M}(p)$. More explicitly it is given by $\widetilde{D}'(p) = D'(p) \circ \epsilon_M$.

$$(3-7) \quad [p : P \rightarrow DM] \mapsto [\widetilde{D}'(p) : M \xrightarrow{\epsilon_M} D'D(M) \xrightarrow{D'(p)} D'P]$$

By Lemma 3.6.(2) $D'(p)$ is an injective cofibration. Since we assume that $\epsilon_M : M \rightarrow S(M) = D'D(M)$ is an injective cofibration, the composite morphism $\widetilde{D}'(p) = D'(p) \circ \epsilon_m$ is an injectively cofibration. By Lemma 3.6(4) the object $D'(P)$ is injectively fibrant in $\mathcal{C}(\mathcal{A})$, hence the object $\widetilde{D}'(p)$ of $\langle \mathcal{J} \rangle_{M/}^{\mathcal{C}}$ belongs to \mathcal{U} .

By abusing notation we denote by \widetilde{D}' the isomorphism of the mapping complexes

$$\widetilde{D}' : \text{Map}_{\mathcal{C}(\mathcal{E})}(P, DM) \rightarrow \text{Map}_{\mathcal{C}(\mathcal{A})}(M, D'P)$$

induced from the above isomorphism $\widetilde{D}'_{P,M}$ of Hom-complexes.

Let $p : P \rightarrow DM$ and $q : Q \rightarrow DM$ be objects of \mathcal{O} . Recall that the mapping complex $\text{Map}_{\mathcal{O}}(p, q)$ is defined by the pull-back diagram

$$(3-8) \quad \begin{array}{ccc} \text{Map}_{\mathcal{O}}(p, q) & \longrightarrow & \text{Map}_{\mathcal{C}(\mathcal{E})}(P, Q) \\ \downarrow & & \downarrow q_* \\ \{p\} & \longrightarrow & \text{Map}_{\mathcal{C}(\mathcal{E})}(P, DM) \end{array}$$

where $q_* := \text{Map}_{\mathcal{C}(\mathcal{E})}(P, q)$. The mapping complex $\text{Map}_{\mathcal{U}}(\widetilde{D}'(q), \widetilde{D}'(p))$ is given by the pull-back diagram

$$(3-9) \quad \begin{array}{ccc} \text{Map}_{\mathcal{U}}(\widetilde{D}'(q), \widetilde{D}'(p)) & \longrightarrow & \text{Map}_{\mathcal{C}(\mathcal{A})}(D'Q, D'P) \\ \downarrow & & \downarrow \widetilde{D}'(q)^* \\ \{\widetilde{D}'(p)\} & \longrightarrow & \text{Map}_{\mathcal{C}(\mathcal{A})}(M, D'P). \end{array}$$

where $\widetilde{D}'(q)^* := \text{Map}_{\mathcal{C}(\mathcal{E})}(\widetilde{D}'(q), D'P)$. It is easy to check that the following diagram is commutative

$$(3-10) \quad \begin{array}{ccc} \text{Map}_{\mathcal{C}(\mathcal{E})}(P, Q) & \xrightarrow{D'} & \text{Map}_{\mathcal{C}(\mathcal{A})}(D'Q, D'P) \\ q_* \downarrow & & \downarrow \widetilde{D}'(q)^* \\ \text{Map}_{\mathcal{C}(\mathcal{E})}(P, DM) & \xrightarrow{\widetilde{D}'} & \text{Map}_{\mathcal{C}(\mathcal{A})}(M, D'P) \\ \uparrow & & \uparrow \\ \{p\} & \xlongequal{\sim} & \{\widetilde{D}'(p)\} \end{array}$$

Therefore the map $D' : \text{Map}_{\mathcal{C}(\mathcal{E})}(P, Q) \rightarrow \text{Map}_{\mathcal{C}(\mathcal{A})}(D'Q, D'P)$ induces the map $\text{Map}_{\mathcal{O}}(p, q) \rightarrow \text{Map}_{\mathcal{U}}(\widetilde{D}'(q), \widetilde{D}'(p))$, which will be denoted by $\widetilde{D}'_{p,q}$. Then we can check that the assignment $\text{Ob } \mathcal{O} \rightarrow \text{Ob } \mathcal{U}, p \mapsto \widetilde{D}'(p)$ and the collection of maps $\{\widetilde{D}'_{p,q}\}$ gives a simplicial functor $\widetilde{D}' : \mathcal{O}^{\text{op}} \rightarrow \mathcal{U}$.

Lemma 3.9. *The simplicial functor \widetilde{D}' is a Dwyer-Kan equivalence.*

Proof. First we show that the functor $\text{Ho}(\widetilde{D}') : \text{Ho}(\mathcal{O}^{\text{op}}) \rightarrow \text{Ho}(\mathcal{U})$ is an essentially surjective. Let $k : M \rightarrow K$ be an object of \mathcal{U} . Then $Dk : DK \rightarrow DM$ is a projective fibration of dg- \mathcal{E} -modules by Lemma 3.6. We take a projective cofibrant replacement $g : P \xrightarrow{\sim} DK$ of DK . Then the composite morphism $p := Dk \circ g : P \rightarrow DM$ belongs to \mathcal{O} . We set $\psi := D'g \circ \epsilon_K$. The following commutative diagram shows that we have a weak equivalence $\psi : k \xrightarrow{\sim} \widetilde{D}'(p)$:

$$\begin{array}{ccccc} M & \xrightarrow{k} & K & & \\ \parallel & & \downarrow \epsilon_K & \searrow \sim \psi & \\ M & \xrightarrow{\epsilon_M} & D'DM & \xrightarrow{D'Dk} & D'DK \xrightarrow{\sim} D'P \\ & & & & \downarrow D'g \end{array}$$

By Lemma 2.2, ψ is a weak homotopy equivalence in \mathcal{U} . Therefore we see the desired result.

We prove that the map $\widetilde{D}'_{p,q}$ of simplicial sets is weak homotopy equivalence. The top arrow $D' : \text{Map}_{\mathcal{C}(\mathcal{E})}(P, Q) \rightarrow \text{Map}_{\mathcal{C}(\mathcal{A})}(D'Q, D'P)$ in the above diagram (3-10) is a weak homotopy equivalence. The bottom arrow $\widetilde{D}' : \text{Map}_{\mathcal{C}(\mathcal{E})}(P, DM) \rightarrow \text{Map}_{\mathcal{C}(\mathcal{A})}(M, D'P)$ in the above diagram (3-10) is an isomorphism of simplicial sets. Since the maps q_* and $\widetilde{D}'(q)^*$ are fibrations of simplicial sets and the standard model structure of sSet is proper, the above diagrams (3-8,3-9) are homotopy pull-back diagrams. Therefore we see the desired result. \square

We denote by Υ the domain functor

$$\Upsilon : \mathcal{O} \rightarrow \mathcal{C}(\mathcal{E}), \quad [p : P \rightarrow DM] \mapsto P.$$

Then we have $\Gamma \circ \widetilde{D}' = D' \circ \Upsilon^{\text{op}}$.

$$\begin{array}{ccc} \mathcal{O}^{\text{op}} & \xrightarrow{\Upsilon^{\text{op}}} & \mathcal{C}(\mathcal{E})^{\text{op}} \\ \widetilde{D}' \downarrow & & \downarrow D' \\ \mathcal{U} & \xrightarrow{\Gamma} & \mathcal{C}(\mathcal{A}) \end{array}$$

We denote by $\Phi : \Upsilon \rightarrow \text{ct}_{\mathcal{O}}(D(M))$ the tautological natural transformation given by the assignment $\Phi : p \mapsto p$. By the construction of \widetilde{D}' , we have the commutative diagram

$$\begin{array}{ccc} D' \circ (\text{ct}_{\mathcal{O}^{\text{op}}} DM) & \xlongequal{\quad} & \text{ct}_{\mathcal{O}^{\text{op}}} S(M) \\ D'(\Phi^{\text{op}}) \downarrow & & \downarrow S(\kappa_{\widetilde{D}'}) \\ D' \circ \Upsilon^{\text{op}} & \xlongequal{\quad} & \Gamma \circ \widetilde{D}' \xrightarrow[\epsilon_{\Gamma}]{} S \circ \Gamma \circ \widetilde{D}'. \end{array}$$

Since the functor \widetilde{D}' is a Dwyer-Kan equivalence, in particular homotopy left cofinal, in view of Lemma 2.16 it is enough to show that the morphism $S(\kappa_{\widetilde{D}'})$ exhibits $S(M)$ as a homotopy limit of $S \circ \Gamma \circ \widetilde{D}'$. By Lemma 2.15 it is enough to show that $D'(\Phi^{\text{op}})$ exhibits $\text{ct}_{\mathcal{O}^{\text{op}}} S(M)$ as a homotopy limit. Since the contravariant simplicial functor $D' = \varepsilon(-, \underline{J})$ preserves quasi-isomorphisms and sends colimit to limit, in view of Lemma 2.17 it is enough to show that the map $\Phi : \Upsilon \rightarrow \text{ct}_{\mathcal{O}}(D(M))$ exhibits $D(M)$ as a homotopy colimit. It is proved in next section Theorem 4.1. We finish the proof of Theorem 3.4.

4 A dg-module is obtained as a tautological filtered homotopy colimit of perfect modules.

In this section we prove the following theorem.

Let \mathcal{A} be a small dg-category and M be a dg \mathcal{A} -module. We equip $\mathcal{C}(\mathcal{A})$ with the projective model structure. We denote by \mathcal{O} the full sub simplicial category $(\text{Perf}_{\text{dg}} \mathcal{A}/M)^{\circ}$ of $\text{Perf}_{\text{dg}} \mathcal{A}/M$. Namely the object $p : P \rightarrow M$ is such that the morphism p of $\mathcal{C}(\mathcal{A})$ is a projective fibration and the domain P of p is a projectively fibrant object which belongs to $\text{Perf}_{\text{dg}} \mathcal{A}$. We denote by $\Upsilon := \Upsilon_M$ the domain functor

$$\Upsilon : \mathcal{O} \rightarrow \mathcal{C}(\mathcal{A}), \quad [p : P \rightarrow M] \mapsto P =: \Upsilon_p$$

We denote by Υ_p the value of Υ at $p \in \mathcal{O}$. Observe that the assignment $\Phi : p \mapsto p$ gives a natural transformation $\Phi : \Upsilon \rightarrow \text{ct}_{\mathcal{O}} M$.

$$\begin{array}{ccc} \Upsilon_p & \xlongequal{\quad} & P \\ \Phi_p \downarrow & & \downarrow p \\ (\text{ct}_{\mathcal{O}} M)_p & \xlongequal{\quad} & M \end{array}$$

Theorem 4.1. *The canonical morphism*

$$\operatorname{hocolim}_{\mathcal{O}} \Upsilon \rightarrow M$$

is a quasi-isomorphism. More precisely the morphism Φ exhibits M as a homotopy colimit of Υ . Namely for any cofibrant replacement $\theta \xrightarrow{\sim} \Phi$ in the functor category $\mathcal{C}(\mathcal{A})^{\mathcal{O}}$ equipped with the projective model structure, the composite map $\Theta \rightarrow \Upsilon \rightarrow \operatorname{ct}_{\mathcal{O}} M$ induces a quasi-isomorphism $\operatorname{colim} \Theta \rightarrow M$.

In the rest of this section we devote to prove this theorem. For $i \in \mathbb{Z}$ and $a \in \mathcal{A}$, taking i -th cohomologies at a , we obtain a natural transformation $H^i(\Phi(a)) : H^i(\Upsilon(a)) \rightarrow H^i(\operatorname{ct}(M)(a)) = H^i(M(a))$ of the functors from \mathcal{O} to $\operatorname{Mod} \mathbf{k}$. We denote by f the morphism $\operatorname{colim} H^i(\Phi(a))$.

$$f := \operatorname{colim}_{\mathcal{O}} H^i(\Phi(a)) : \operatorname{colim}_{\mathcal{O}} H^i(\Upsilon(a)) \rightarrow H^i(M(a))$$

Lemma 4.2. *The morphism f is an isomorphism.*

We set $\alpha := a^\wedge[-i]$. Recall that by Yoneda Lemma, for a dg \mathcal{A} -module N , there exists a natural morphism

$$q := q^N : Z^i(N(a)) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{C}(\mathcal{A})}(\alpha, N).$$

We denote by u the element of $Z^i(\alpha(a))$ such that $q_u = 1_\alpha$. Then the morphism $q_n : \alpha \rightarrow N$ in $\mathcal{C}(\mathcal{A})$ is characterized by the property that $Z^i(q_n(a))(u) = n$.

Proof. We set $L := \operatorname{colim}_{\mathcal{O}} H^i(\Upsilon(a))$. First we prove that f is surjective. Let \bar{m} be an element of $H^i(M(a))$. We choose a representative element m in $Z^i(M(a))$ of \bar{m} . We decompose $q_m : \alpha \rightarrow M$ into a trivial cofibration $s : \alpha \xrightarrow{\sim} \beta$ followed by a fibration $p : \beta \rightarrow M$. Since α is cofibrant, β is also cofibrant. Hence p is an object of \mathcal{O} .

$$\begin{array}{ccc} \alpha & \xrightarrow{\sim s} & \beta \\ & \searrow q_m & \downarrow p \\ & & M \end{array}$$

Now the following commutative diagram tells us that m is in the image of f

$$\begin{array}{ccccc} H^i(\alpha(a)) & \xrightarrow{\cong} & H^i(\beta(a)) & \xrightarrow{\quad} & L \\ & \searrow H^i(q_m(a)) & \downarrow H^i(p(a)) & \nearrow f & \\ & & H^i(M(a)) & & \end{array}$$

Next we prove that f is injective. Let $\bar{\xi}_1, \bar{\xi}_2$ be elements of L such that $f(\bar{\xi}_1) = f(\bar{\xi}_2)$. For $s = 1, 2$ there exists object $p_s : P_s \rightarrow M$ of \mathcal{O} such that there exists an element \bar{x}_s of $H^i(P_s(a))$ such that $\eta_s(\bar{x}_s) = \bar{\xi}_s$ where η_s is the canonical morphism $H^i(P_s(a)) = H^i(\Upsilon_{p_s}(a)) \rightarrow L$. We choose a representative x_s in $Z^i(P_s(a))$ of \bar{x}_s for $s = 1, 2$. Then the morphisms $p_1 \circ q_{x_1}$ and $p_2 \circ q_{x_2}$ are homotopic. Therefore we have the following commutative diagram

$$\begin{array}{ccccc} & \alpha & & \alpha & \\ q_{x_1} \swarrow & & \lambda_1 \searrow & \lambda_2 \swarrow & q_{x_2} \searrow \\ P_1 & & \operatorname{Cyl}(\alpha) & & P_2 \\ & p_1 \searrow & \downarrow H & \swarrow p_2 & \\ & & M & & \end{array}$$

where $\text{Cyl}(\alpha)$ is a cylinder object of α , e.g., $\text{Cyl}(\alpha) = \alpha \otimes \Delta^1$. In other words, we have the following diagram in $\text{Perf}_{\text{dg}}\mathcal{A}/M$

$$\begin{array}{ccccc} & \mu_1 & & \mu_2 & \\ q_{x_1} \swarrow & & \lambda_1 \searrow & \lambda_2 \swarrow & q_{x_2} \searrow \\ p_1 & & H & & p_2 \end{array}$$

where we set $\mu_s := p_s \circ q_{x_s} = H \circ \lambda_s$ for $s = 1, 2$. Using the following Lemma 4.3 we obtain the following homotopy commutative diagram in $\text{Perf}_{\text{dg}}\mathcal{A}/M$ such that r_1, r_2, r_3 belong to \mathcal{O}

$$\begin{array}{ccccc} & \mu_1 & & \mu_2 & \\ q_{x_1} \swarrow & & \lambda_1 \searrow & \lambda_2 \swarrow & q_{x_2} \searrow \\ p_1 & & H & & p_2 \\ & \searrow & & \swarrow & \\ & r_1 & & r_2 & \\ & \searrow & & \swarrow & \\ & r_3 & & & \end{array}$$

Applying domain functor and taking cohomology groups to this diagram, we obtain the following strictly commutative diagram in $\text{Mod } \mathbf{k}$

$$\begin{array}{ccccc} H^i(\alpha(a)) & \xlongequal{\quad} & H^i(\alpha(a)) & & \\ \swarrow & & \searrow & & \swarrow \searrow \\ H^i(\Upsilon_{p_1}(a)) & & H^i(\text{Cyl}(\alpha)(a)) & & H^i(\Upsilon_{p_2}(a)) \\ \searrow & \swarrow & \searrow & \swarrow & \searrow \\ H^i(\Upsilon_{r_1}(a)) & & & & H^i(\Upsilon_{r_2}(a)) \\ & \searrow & & \swarrow & \\ & H^i(\Upsilon_{r_3}(a)) & & & \end{array}$$

Chasing the homotopy class of the element $u \in Z^i(\alpha(a))$, we see that $\bar{\xi}_1 = \eta_1(\bar{x}_1) = \eta_2(\bar{x}_2) = \bar{\xi}_2$. \square

We denote by $(\text{Perf}_{\text{dg}}\mathcal{A}^{\text{p-cof}})_{/M}$ the subcategory of $\text{Perf}_{\text{dg}}\mathcal{A}/M$ consisting those objects $p : P \rightarrow M$ such that the domain P is projectively cofibrant perfect \mathcal{A} -module.

Lemma 4.3. (1) Let $p : P \rightarrow M$ be an object of $(\text{Perf}_{\text{dg}}\mathcal{A}^{\text{p-cof}})_{/M}$ and $q : Q \rightarrow M$ a object of \mathcal{O} . Then there exists a diagram $p \rightarrow r \leftarrow q$ in $(\text{Perf}_{\text{dg}}\mathcal{A}^{\text{p-cof}})_{/M}$ such that r belongs to \mathcal{O} .

(2) Let $p : P \rightarrow M$ be an object of $(\text{Perf}_{\text{dg}}\mathcal{A}^{\text{p-cof}})_{/M}$ and $q : Q \rightarrow M$ a object of \mathcal{O} . For any parallel two morphisms $f, g : p \rightarrow q$ in \mathcal{O} there exists an morphism $h : q \rightarrow r$ such that the composite morphisms $h \circ f$ and $h \circ g$ are homotopic.

Proof. (1) It is enough to set $R = P \oplus Q$ and $r = (p, q)$.

$$\begin{array}{ccccc} P & \xrightarrow{t(1,0)} & P \oplus Q & \xleftarrow{t(0,1)} & Q \\ & \searrow p & \downarrow (p,q) & \swarrow q & \\ & & M & & \end{array}$$

(2) follows from Lemma 2.13 and the following remark. \square

Let $\mathbf{rep} : \Theta \xrightarrow{\sim} \Upsilon$ be a cofibrant replacement of Υ . Then we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{colim}_{\mathcal{O}} H^i(\Theta(a)) & \xrightarrow{\mathbf{can}} & H^i(\mathrm{colim}_{\mathcal{O}} \Theta(a)) \\ \mathrm{colim}_{\mathcal{O}} H^i(\mathbf{rep}) \downarrow & & \downarrow H^i(\mathrm{colim}_{\mathcal{O}} \Phi \circ \mathbf{rep}) \\ \mathrm{colim}_{\mathcal{O}} H^i(\Upsilon(a)) & \xrightarrow{\mathrm{colim}_{\mathcal{O}} H^i(\Phi(a))} & H^i(M(a)) \end{array}$$

where \mathbf{can} is the canonical morphism. We need to show that the left vertical arrow is an isomorphism. By Lemma 4.2 the bottom arrow is an isomorphism. Since the morphism \mathbf{rep} is a weak equivalence, the right vertical arrow is also an isomorphism. Therefore we only have to show that the top arrow \mathbf{can} is an isomorphism. It follows from the following two lemmas: the ∞ -category $N(\mathcal{O})$ is filtered (Lemma 4.4) and filtered homotopy colimit commutes with taking cohomology group (Lemma 4.5).

Lemma 4.4. *The ∞ -category $N(\mathcal{O})$ is filtered.*

Proof. Since \mathcal{O} is a fibrant simplicial category, by [20, Proposition 5.3.1.13, Definition 5.3.1.1] it is enough to prove the following two conditions:

- (1) For every finite set $\{p_i\}_{i=1}^n$ of objects of \mathcal{O} , there exists an objects $p \in \mathcal{O}$ and morphisms $\phi_i : p_i \rightarrow p$.
- (2) For every pair p, q of objects of \mathcal{O} , every finite simplicial set X and every morphism $X \rightarrow \mathrm{Map}_{\mathcal{O}}(p, q)$ there exists a morphism $q \rightarrow r$ such that the induced map $X \rightarrow \mathrm{Map}_{\mathcal{O}}(p, r)$ is null homotopic.

(1) For $p_i : P_i \rightarrow M$, $i = 1, 2, \dots, n$, it is enough to set $P := (P_1, \dots, P_n) : \bigoplus_{i=1}^n P_i \rightarrow M$ and $\phi_i : P_i \rightarrow P$ be the morphism in \mathcal{O} induced from the i -th canonical injection $P_i \rightarrow \bigoplus_{i=1}^n P_i$.

(2) For simplicity we denote by $[-, +] = \mathrm{Hom}_{\mathbf{hSet}}(-, +)$ the Hom set of the homotopy category \mathbf{hSet} .

Let f be an element of $\mathrm{Hom}_{\mathbf{sSet}}(X, \mathrm{Map}_{\mathcal{O}}(p, q))$. We denote by the same symbol $f \in [X, \mathrm{Map}_{\mathcal{O}}(p, q)]$ the homotopy class of f . The problem is to show that there exists a morphism $g : q \rightarrow r$ such that the image of f of the morphism $[X, \mathrm{Map}_{\mathcal{O}}(p, q)] \rightarrow [X, \mathrm{Map}_{\mathcal{O}}(p, r)]$ induced from g belongs to the image of the morphism $[\ast, \mathrm{Map}_{\mathcal{O}}(p, r)] \rightarrow [X, \mathrm{Map}_{\mathcal{O}}(p, r)]$ induced from a unique map $X \rightarrow \ast$.

Recall that we have a natural isomorphism

$$[X, \mathrm{Map}_{\mathcal{O}}(p, q)] \cong \mathrm{Hom}_{\mathbf{h}\mathcal{O}}(p \otimes X, q).$$

Therefore the problem is equivalent to show that there exists a morphism $g : q \rightarrow r$ in \mathcal{O} such that we have a homotopy commutative diagram

$$\begin{array}{ccc} p \otimes X & \xrightarrow{f} & q \\ \alpha \downarrow & & \downarrow g \\ p & \longrightarrow & r \end{array}$$

where $\alpha : p \otimes X \rightarrow p$ is the morphism induced from a unique morphism $X \rightarrow \ast$. However this is a consequence of Lemma 2.13 and the following remark. \square

Lemma 4.5. *Let V be a filtered ∞ -category and $F : V \rightarrow N(\mathcal{C}(\mathcal{A})^\circ)$ an ∞ -functor. Then the canonical morphism*

$$\mathrm{colim}_V H^i(F(a)) \rightarrow H^i(\mathrm{hocolim}_V F(a))$$

is an isomorphism for $i \in \mathbb{Z}$ and $a \in \mathcal{A}$.

Proof. By [20, Proposition 5.3.16] there exists a filtered partially ordered set A and a cofinal map $\iota : N(A) \rightarrow V$. Therefore we may replace V with $N(A)$ and F with $F \circ \iota$. Since we have an isomorphism

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{sCat})}(A, \mathcal{C}(\mathcal{A})^\circ) \cong \mathrm{Hom}_{\mathrm{Ho}(\infty\mathrm{Cat})}(N(A), N(\mathcal{C}(\mathcal{A})^\circ)),$$

where we denote by $\infty\mathrm{Cat}$ the model category sSet equipped with the Joyal model structure, we may replace $F \circ \iota$ with $N(f)$ for some functor $f : A \rightarrow \mathcal{C}(\mathcal{A})$. Now the result follows from the well-known fact that filtered colimit commute with taking cohomology group. \square

This finish the proof of Theorem 4.1.

5 Miscellaneous results

5.1 On the choice in Construction 3.1.(III)

In this subsection we will show that the conclusion of Theorem 3.4 is independent to the choice in Construction 3.1.(III). More precisely, in Construction 3.1.(III) we take a different choice $\underline{\mathcal{J}}_2 \xrightarrow{\sim} \underline{\mathcal{J}}'$ of a fibrant replacement and denote by S' the corresponding bi-duality functor.

Proposition 5.1. *The canonical morphism $S(\kappa) : ctS(M) \rightarrow S \circ \Gamma$ exhibits $S(M)$ as a homotopy limit of $S \circ \Gamma$ if and only if the canonical morphism $S'(\kappa) : ctS'(M) \rightarrow S' \circ \Gamma$ exhibits $S'(M)$ as a homotopy limit of $S' \circ \Gamma$.*

Corollary 3.7 claim that the quasi-isomorphism class of $S(M)$ is independent to the choice of $\underline{\mathcal{J}}$ in the step (III). However we need finer information for our purpose. For latter reference, we consider the following more general situation.

Let \mathcal{F} be a locally cofibrant dg-category. Assume that there exists a quasi-equivalence $g : \mathcal{F} \xrightarrow{\sim} \mathcal{E}$. To prove Proposition 5.1 it is enough to set $\mathcal{F} = \mathcal{E}$ and $g = 1_{\mathcal{E}}$. By Lemma 2.8 the dg-functor $\tilde{g} := 1_{\mathcal{A}} \otimes g$ is a quasi-equivalence from $\mathcal{A} \otimes \mathcal{F}$ to $\mathcal{A} \otimes \mathcal{E}$. Note that for $\mathcal{A} \otimes \mathcal{E}$ -module X and for \mathcal{A} -module N , we have natural isomorphisms

$$(5-11) \quad g_*(X(a \otimes -)) \cong (\tilde{g}_*X)(a \otimes -), \quad g_{*\mathcal{A}}(N, X) \cong {}_{\mathcal{A}}(N, \tilde{g}_*X).$$

We assume that the fibrant replacement $\underline{\mathcal{J}}$ of the diagonal module $\underline{\mathcal{J}}_2$ is taken by a injectively trivial cofibration $\underline{\mathcal{J}}_2 \xrightarrow{\sim} \underline{\mathcal{J}}$. Then the induced morphism $\tilde{g}_*\underline{\mathcal{J}}_2 \rightarrow \tilde{g}_*\underline{\mathcal{J}}$ is also injectively trivial cofibration. Let $\tilde{g}_*\underline{\mathcal{J}}_2 \xrightarrow{\sim} \underline{\mathcal{J}}'$ be a fibrant replacement. Then by the lifting property, there exists a quasi-isomorphism $\alpha : \tilde{g}_*\underline{\mathcal{J}} \rightarrow \underline{\mathcal{J}}'$.

We define the endofunctors S' and T of $\mathcal{C}(\mathcal{A})$ to be

$$S'(-) := {}_{\mathcal{F}}(\mathcal{A}(-, \underline{\mathcal{J}}'), \underline{\mathcal{J}}'), \quad T(-) := {}_{\mathcal{F}}(\mathcal{A}(-, \tilde{g}_*\underline{\mathcal{J}}), \underline{\mathcal{J}}').$$

We denote by $\epsilon : 1_{\mathcal{C}(\mathcal{A})} \rightarrow S'$ the natural transformation induced from the evaluation map. Then as in the proof of Lemma 3.8, the morphism α induces the following commutative diagram of endofunctors of $\mathcal{C}(\mathcal{A})$

$$\begin{array}{ccc} 1_{\mathcal{C}(\mathcal{A})} & \xrightarrow{\epsilon} & S \\ \epsilon' \downarrow & & \downarrow \rho \\ S' & \xrightarrow{\lambda} & T \end{array}$$

We claim that the morphisms λ, ρ are weak equivalences of endofunctors. Namely, for $N \in \mathcal{C}(\mathcal{A})$ the morphisms λ_N, ρ_N are quasi-isomorphisms. As in the same way that we prove ψ_1 is quasi-isomorphism

in the proof of Lemma 3.8.(1), we can prove that λ_N is a quasi-isomorphism. The morphism ρ_N evaluated at $a \in \mathcal{A}$ is given in the following way

$$\begin{aligned} \varepsilon(\mathcal{A}(N, \underline{\mathcal{J}}), \underline{\mathcal{J}}(a \otimes -)) &\xrightarrow{g_*} \mathcal{F}(g_{*\mathcal{A}}(N, \underline{\mathcal{J}}), g_*(\underline{\mathcal{J}}(a \otimes -))) \cong \mathcal{F}(\mathcal{A}(N, \tilde{g}_* \underline{\mathcal{J}}), (\tilde{g}_* \underline{\mathcal{J}})(a \otimes -)) \\ &\xrightarrow{\tilde{\alpha}_a} \mathcal{F}(\mathcal{A}(N, \tilde{g}_* \underline{\mathcal{J}}), \underline{\mathcal{J}}'(a \otimes -)). \end{aligned}$$

The last morphism $\tilde{\alpha}_a$ is induced from the quasi-isomorphism $\alpha(a) : (\tilde{g}_* \underline{\mathcal{J}})(a \otimes -) \xrightarrow{\sim} \underline{\mathcal{J}}'(a \otimes -)$. Since $\underline{\mathcal{J}}(a \otimes -)$ is injectively fibrant, the Hom complex $\varepsilon(\mathcal{A}(N, \underline{\mathcal{J}}), \underline{\mathcal{J}}(a \otimes -))$ compute the derived Hom complex on the top of the following diagram.

$$\begin{array}{c} \mathbb{R} \operatorname{Hom}_{\mathcal{E}}(\mathcal{A}(N, \underline{\mathcal{J}}), \underline{\mathcal{J}}(a \otimes -)) \\ \downarrow g_* \\ \mathbb{R} \operatorname{Hom}_{\mathcal{F}}(g_{*\mathcal{A}}(N, \underline{\mathcal{J}}), g_*(\underline{\mathcal{J}}(a \otimes -))). \end{array}$$

On the other hands, the morphism $\alpha(a)$ is a fibrant replacement of $(\tilde{g}_* \underline{\mathcal{J}})(a \otimes -)$ by Lemma 2.11. Hence the Hom complex $\mathcal{F}(\mathcal{A}(M, \tilde{g}_* \underline{\mathcal{J}}), \underline{\mathcal{J}}'(a \otimes -))$ compute the derived Hom complex in the bottom of above diagram. Observe that the morphism ρ_N is quasi-isomorphic to the morphism between the above derived Hom complexes induced from g_* . By Lemma 2.7 the restriction functor $g_* : \mathcal{D}(\mathcal{E}) \rightarrow \mathcal{D}(\mathcal{F})$ gives an equivalence. Hence we conclude that ρ_N is quasi-isomorphism.

Compositing the above diagram of endofunctors of $\mathcal{C}(\mathcal{A})$ with the morphism $\kappa : \operatorname{ct}_{\mathcal{U}} M \rightarrow \Gamma$, we obtain the commutative diagram

$$\begin{array}{ccccc} \operatorname{ct}_{\mathcal{U}} S'(M) & \xrightarrow{\sim} & \operatorname{ct}_{\mathcal{U}} T(M) & \xleftarrow{\sim} & \operatorname{ct}_{\mathcal{U}} S(M) \\ S'(\kappa) \downarrow & & T(\kappa) \downarrow & & \downarrow S(\kappa) \\ S' \circ \Gamma & \xrightarrow{\sim} & T \circ \Gamma & \xleftarrow{\sim} & S \circ \Gamma. \end{array}$$

By Lemma 2.15, we see that $S(\kappa)$ exhibits $S(M)$ as a homotopy limit of $S \circ \Gamma$ if and only if $S'(\kappa)$ exhibits $S'(M)$ as a homotopy limit of $S' \circ \Gamma$.

We define a dg-category \mathcal{B} to be

$$\operatorname{Ob}(\mathcal{B}) := \operatorname{Ob}\mathcal{A}, \quad \mathcal{B}(a, b) := \mathcal{F}(\underline{\mathcal{J}}'(b \otimes -), \underline{\mathcal{J}}'(a \otimes -)).$$

The identity on the objects induces a dg-functor $\tilde{\iota} : \mathcal{A} \rightarrow \mathcal{B}$. Note that by [5, Lemma 3.2] in the homotopy category $\operatorname{Ho}(\operatorname{dgCat}_{\mathcal{A}})$ the object $\iota : \mathcal{A} \rightarrow \mathbb{B}\operatorname{ic}_{\mathcal{A}}(\mathcal{J})$ is isomorphic to $\tilde{\iota} : \mathcal{A} \rightarrow \mathcal{B}$. In other words, we can compute the derived bi-commutator category $\mathbb{B}\operatorname{ic}_{\mathcal{A}}(\mathcal{J})$ by using \mathcal{F} and $\underline{\mathcal{J}}'$.

5.2 Derived bi-commutators over equivalent subcategories

Proposition 5.2. *Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a dg-functor and $\mathcal{J} \subset \mathcal{D}(\mathcal{B})$ a small subcategory. Assume that the restriction functor $f_* : \mathcal{D}(\mathcal{B}) \rightarrow \mathcal{D}(\mathcal{A})$ induces an equivalence between \mathcal{J} and $f_*\mathcal{J}$. Then in the homotopy category $\operatorname{Ho}(\operatorname{dgCat})$ we have a quasi-fully faithful functor $f^\heartsuit : \mathbb{B}\operatorname{ic}_{\mathcal{A}}(f_*\mathcal{J}) \rightarrow \mathbb{B}\operatorname{ic}_{\mathcal{B}}(\mathcal{J})$ and the commutative diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ \downarrow \iota_{f_*\mathcal{J}} & & \downarrow \iota_{\mathcal{J}} \\ \mathbb{B}\operatorname{ic}_{\mathcal{A}}(f_*\mathcal{J}) & \xrightarrow{f^\heartsuit} & \mathbb{B}\operatorname{ic}_{\mathcal{B}}(\mathcal{J}). \end{array}$$

Remark 5.3. Since the canonical morphisms $\iota_{f_*\mathcal{J}}$ and $\iota_{\mathcal{J}}$ induces identities on objects, if the functor f is a quasi-essentially surjective, then the functor f^\heartsuit gives a quasi-equivalence of the derived bi-commutator categories.

We may assume that the functor $f : \mathcal{A} \rightarrow \mathcal{B}$ is a cofibration. We construct $\underline{\mathcal{J}}_1 \subset \mathcal{C}(\mathcal{B})$ and $\mathcal{F} := \mathcal{E}nd_{\mathcal{A}}(\underline{\mathcal{J}}_1)^{op}$ by using Construction 3.1. By Lemma 2.10, the \mathcal{A} -module $f_* \underline{\mathcal{J}}_1$ is injectively fibrant. Hence the dg-categories $\mathcal{E} := \mathcal{E}nd_{\mathcal{A}}(f_* \underline{\mathcal{J}}_1)^{op}$ is locally cofibrant. By assumption the dg-functor $g : \mathcal{F} \rightarrow \mathcal{E}$ induced from f_* is a quasi-equivalence of locally cofibrant dg-categories.

Let $\underline{\mathcal{J}}_2$ and $f_* \underline{\mathcal{J}}_2$ be the diagonal modules of $\underline{\mathcal{J}}_1$ and $f_* \underline{\mathcal{J}}$ respectively. We set $\tilde{f} := f \otimes 1_{\mathcal{F}} : \mathcal{A} \otimes \mathcal{F} \rightarrow \mathcal{B} \otimes \mathcal{F}$ and $\tilde{g} := 1_{\mathcal{A}} \otimes g : \mathcal{A} \otimes \mathcal{F} \rightarrow \mathcal{A} \otimes \mathcal{E}$. Then we can check an isomorphism $\tilde{f}_* \underline{\mathcal{J}}_2 \cong \tilde{g}_*(f_* \underline{\mathcal{J}}_2)$ of $\mathcal{A} \otimes \mathcal{F}$ -modules. Let $\underline{\mathcal{J}}_2 \xrightarrow{\sim} \underline{\mathcal{J}}$ be an injectively fibrant replacement of $\underline{\mathcal{J}}_2$. By Lemma 2.10 $\tilde{f}_* \underline{\mathcal{J}}$ is an injective fibrant replacement of $\tilde{f}_* \underline{\mathcal{J}}_2$. Therefore according to the consideration in Section 5.1 we can compute the derived bi-dual over $f_* \mathcal{J} \in \mathcal{D}(\mathcal{A})$ by using $\tilde{f}_* \underline{\mathcal{J}}$.

$$S_{f_* \mathcal{J}}(-) := {}_{\mathcal{F}}(\mathcal{A}(-, \tilde{f}_* \underline{\mathcal{J}}), \tilde{f}_* \underline{\mathcal{J}}).$$

The restriction functor $f_* : \mathcal{C}(\mathcal{B}) \rightarrow \mathcal{C}(\mathcal{A})$ induces the morphism ${}_{\mathcal{B}}(f_* M, \underline{\mathcal{J}}) \rightarrow {}_{\mathcal{A}}(f_* f^* M, \tilde{f}_* \underline{\mathcal{J}})$ of dg \mathcal{E} -modules. We denote by $\alpha_M : S_{f_* \mathcal{J}}(f_* f^* M) \rightarrow f_* S_{\mathcal{J}}(f^* M)$ which is induced from the above morphism

$$\alpha_M : S_{f_* \mathcal{J}}(f_* f^* M) = {}_{\mathcal{F}}(\mathcal{A}(f_* f^* M, \tilde{f}_* \underline{\mathcal{J}}), \tilde{f}_* \underline{\mathcal{J}}) \rightarrow {}_{\mathcal{F}}(\mathcal{B}(f^* M, \underline{\mathcal{J}}), \tilde{f}_* \underline{\mathcal{J}}) \cong f_* S_{\mathcal{J}}(f^* M).$$

where for the last isomorphism we use an isomorphism (5-11). Let $\eta_M : M \rightarrow f_* f^* M$ be the unit map. Then we have the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\eta_M} & f_* f^* M \\ \epsilon_M \downarrow & & \downarrow \epsilon_{f_* f^* M} \\ S_{f_* \mathcal{J}}(M) & \xrightarrow{S_{f_* \mathcal{J}}(\eta_M)} & S_{f_* \mathcal{J}}(f_* f^* M) \\ & \searrow \beta_M & \downarrow \alpha_M \\ & & f_* S_{\mathcal{J}}(f^* M) \end{array} \quad \begin{array}{c} \curvearrowright \\ f_* \epsilon_{f_* M} \end{array}$$

where we denote by β_M the composition $\beta_M := \alpha_M \circ S_{f_* \mathcal{J}}(\eta_M)$. However the morphism β_M is nothing but the morphism induced from the adjunction isomorphism ${}_{\mathcal{A}}(M, \tilde{f}_* \underline{\mathcal{J}}) \cong {}_{\mathcal{B}}(f^* M, \underline{\mathcal{J}})$. In particular it is an isomorphism.

In the case when $M = a^\wedge$ for $a \in \mathcal{A}$, the morphism $\beta_{a^\wedge}(b)$ gives an isomorphism

$$S_{f_* \mathcal{J}}(a^\wedge)(b) = {}_{\mathcal{E}}(\tilde{f}_* \underline{\mathcal{J}}(b), \tilde{f}_* \underline{\mathcal{J}}(a)) \cong {}_{\mathcal{E}}(\underline{\mathcal{J}}(fb), \underline{\mathcal{J}}(fa)) = f_* S_{\mathcal{J}}(f^* a^\wedge)(b)$$

of dg \mathbf{k} -modules for $b \in \mathcal{A}$. The family of the morphisms β_{a^\wedge} with $a \in \mathcal{A}$ induces a fully faithful dg-functor $f^\heartsuit : \mathcal{B}ic_{\mathcal{A}}(f_* \mathcal{J}) \hookrightarrow \mathcal{B}ic_{\mathcal{B}}(\mathcal{J})$. It is straightforward to check that we have the commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ \downarrow \iota_{f_* \mathcal{J}} & & \downarrow \iota_{\mathcal{J}} \\ \mathcal{B}ic_{\mathcal{A}}(f_* \underline{\mathcal{J}}) & \xrightarrow{f^\heartsuit} & \mathcal{B}ic_{\mathcal{B}}(\underline{\mathcal{J}}). \end{array}$$

6 Completion via derived bi-commutator

6.1 Completion theorems

Theorem 6.1. *Let R be a ring and \mathfrak{a} a two-sided ideal of R such that the canonical functor $\mathcal{D}(\mathfrak{a}\text{-tor}) \rightarrow \mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$ gives an equivalence. Assume that R/\mathfrak{a}^n belongs to $\langle R/\mathfrak{a} \rangle^{\mathcal{D}}$ for $n \geq 0$. (e.g.*

R/\mathfrak{a} has finite global dimension and R is Noetherian.) We denote by \widehat{R} the \mathfrak{a} -adic completion. Then in the homotopy category $\mathrm{Ho}(\mathrm{dgCat})$, there exists an isomorphism $\mathbb{B}\mathrm{ic}_R(R/\mathfrak{a}) \cong \widehat{R}$ which fits into the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\iota} & \mathbb{B}\mathrm{ic}_R(R/\mathfrak{a}) \\ & \searrow \text{comp} & \parallel \\ & & \widehat{R} \end{array}$$

where the slant arrow $\text{comp} : R \rightarrow \widehat{R}$ is the canonical completion morphism.

A proof is given in the next section 6.2.

Theorem 6.2. *Let R be a ring and \mathfrak{a} an two-sided ideal such that the canonical functor $\mathcal{D}(\mathfrak{a}\text{-tor}) \rightarrow \mathcal{D}_{\mathfrak{a}\text{-tor}}(\mathcal{R})$ gives an equivalence. Let K be a compact generator of $\mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$. Then in the homotopy category $\mathrm{Ho}(\mathrm{dgCat})$, there exists an isomorphism $\mathbb{B}\mathrm{ic}_R(K) \cong \widehat{R}$ which fits into the commutative diagram*

$$\begin{array}{ccc} R & \xrightarrow{\iota} & \mathbb{B}\mathrm{ic}_R(K) \\ & \searrow \text{comp} & \parallel \\ & & \widehat{R} \end{array}$$

where the slant arrow $\text{comp} : R \rightarrow \widehat{R}$ is the canonical completion morphism.

This theorem is proved in the same way of the proof of Theorem 6.1 by the following Lemma 6.3.

Let R be a ring. Let K be an object of $\mathcal{C}(R)$. We denote by $\overline{\langle K \rangle}^{\mathcal{D}}$ the smallest localizing subcategory of $\mathcal{D}(R)$ containing K . We denote by $\overline{\langle K \rangle}^{\mathcal{C}}$ the pre-image of $\overline{\langle K \rangle}^{\mathcal{D}}$ by the homotopy functor $\mathcal{C}(R) \rightarrow \mathcal{D}(R)$. We set $\mathcal{U} := (\overline{\langle K \rangle}^{\mathcal{C}})^{\circ}$ and $\overline{\mathcal{U}} := \left(\overline{\langle K \rangle}_{R/}^{\mathcal{C}} \right)^{\circ}$. Let $\overline{\Gamma} : \overline{\mathcal{U}} \rightarrow \mathcal{C}(\mathcal{A})$ and $\Gamma : \mathcal{U} \rightarrow \mathcal{C}(\mathcal{A})$ be the co-domain functors. Then the restriction $\overline{\Gamma}|_{\mathcal{U}}$ to \mathcal{U} equals to Γ .

Lemma 6.3. *Assume that K is compact in $\overline{\langle K \rangle}^{\mathcal{D}}$. Then the canonical map $\mathrm{ct}_{\overline{\mathcal{U}}} S(R) \rightarrow S \circ \overline{\Gamma}$ exhibits $S(R)$ as a homotopy limit of $S \circ \overline{\Gamma}$.*

Proof. It is enough to prove that \mathcal{U} is a left homotopy co-final subcategory of $\overline{\mathcal{U}}$. In the same way of the proof of Lemma 4.4 we prove that the simplicial category $\left(\overline{\langle K \rangle}_{R/}^{\mathcal{C}} \right)^{\circ}$ is homotopy left filtered. Therefore by Lemma 2.20 and Lemma 2.1, it is enough to show that for any morphism $\overline{\ell} : R \rightarrow \overline{L}$ in $\mathcal{D}(R)$ with $\overline{L} \in \overline{\langle K \rangle}^{\mathcal{D}}$ there exists a factorization $\overline{\ell} = \psi \circ \ell$ such that $\ell : R \rightarrow L$ is a morphism with $L \in \langle K \rangle^{\mathcal{D}}$.

We take a projective cofibrant replacement $\tilde{K} \xrightarrow{\sim} K$. We set $S := \mathrm{End}_R^{\bullet}(\tilde{K})^{\mathrm{op}}$. Then by [28, Theorem 3.5] the adjoint pair

$$- \otimes_S \tilde{K} : \mathcal{C}(S) \rightleftarrows \mathcal{C}(R) : {}_R(\tilde{K}, -)$$

induces an equivalence between $\mathcal{D}(S)$ and $\overline{\langle K \rangle}^{\mathcal{D}}$. Therefore \overline{L} is quasi-isomorphic to a dg R -module of form $P \otimes_S \tilde{K}$ for some semi-free dg S -module P . Therefore we may assume that the underlying complex $\overline{L}^{\#}$ of \overline{L} is a direct sum of a direct summand of modules of the form $\tilde{K}[a]$, $a \in \mathbb{Z}$ and the differential $d_{\overline{L}}$ is given by one-sided twist. Now we can easily see that for any cocycle ℓ of \overline{L} there exists a sub-complex L of \overline{L} such that $\ell \in L$ and $L \in \langle K \rangle$. \square

It is proved by [27, Corollary 3.31] that if a ring R is commutative and an ideal \mathfrak{a} is weakly pro-regular, then the canonical functor $\mathcal{D}(\mathfrak{a}\text{-tor}) \rightarrow \mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$ gives an equivalence. Therefore as a corollary of Theorem 6.2, we reprove [28, Theorem 4.2].

Corollary 6.4. *Let R be a commutative ring and \mathfrak{a} a weakly pro-regular ideal. Let K be a compact generator of $\mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$. Then we have the following quasi-isomorphism of dg-algebras under R .*

$$\mathbb{B}ic_R(K) \simeq \widehat{R}.$$

The condition that the canonical functor $\mathcal{D}(\mathfrak{a}\text{-tor}) \rightarrow \mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$ gives an equivalence is satisfied if the subcategory $\mathfrak{a}\text{-tor}$ is closed under taking injective hull. This condition is satisfied if a right ideal \mathfrak{a} has the Artin-Rees property.

Let R be a Noetherian algebra. Namely the center $Z = \text{Cen}(R)$ is a Noetherian ring and R is a finitely generated as Z -module. If a two-sided ideal \mathfrak{a} is of form $\tilde{\mathfrak{a}}R$ for some ideal $\tilde{\mathfrak{a}}$ of Z , then it has the Artin-Rees property. Let $\{a_1, \dots, a_n\}$ be a generator of the ideal $\tilde{\mathfrak{a}}$ of Z . It is also a generator of the ideal \mathfrak{a} of R . Then we can prove that the Koszul complex $K = K(R; a_1, \dots, a_n)$ is a compact generator of $\mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$ by the same method of the proof of [1, Proposition 6.1]. Therefore we obtain the following corollary.

Corollary 6.5. *Let R , \mathfrak{a} and K be as above. Then we have the following quasi-isomorphism of dg-algebras under R .*

$$\mathbb{B}ic_R(K) \simeq \widehat{R}.$$

6.2 A proof of Theorem 6.1.

We set $\mathcal{I} := (\mathbb{Z}_{\geq 1})^{\text{op}}$. Let $\tilde{\omega}^n : R \rightarrow R/\mathfrak{a}^n$ be the canonical projection for $n \geq 1$ and $\psi^{m,n} : R/\mathfrak{a}^m \rightarrow R/\mathfrak{a}^n$ the canonical projection for $m \geq n$. We take a cofibrant replacement $\rho : R' \xrightarrow{\sim} R$ of R . We denote by $\Phi : \mathcal{I} \rightarrow \text{dga}$ the functor which sends $n \geq 1$ to R/\mathfrak{a}^n and $m \geq n$ to the canonical projection $R/\mathfrak{a}^m \rightarrow R/\mathfrak{a}^n$. The family of maps $\tilde{\omega}^n \circ \rho : R' \rightarrow R/\mathfrak{a}^n$ induces the morphism $\text{ct}_{\mathcal{I}}(R') \rightarrow \Phi$ in the functor category $\text{dga}^{\mathcal{I}}$. We decompose this morphism into a cofibration $\varpi^\bullet : \text{ct}_{\mathcal{I}}(R') \rightarrow \Phi'$ followed by a trivial fibration $\Phi' \xrightarrow{\sim} \Phi$. Note that since the canonical projection $R/\mathfrak{a}^{n+1} \rightarrow R/\mathfrak{a}^n$ is fibration, we may assume that $\Phi'(\text{geq}^{m,n})$ is also fibration. By abusing notation we denote by the same symbol Φ' the functor $\mathcal{I} \rightarrow \mathcal{C}(R')$ which sends n to Φ'_n . We take a fibrant replacement $\lambda : \Phi' \xrightarrow{\sim} \Phi''$ in the functor category $\mathcal{C}(R')^{\mathcal{I}}$ with the injective model structure where the category $\mathcal{C}(R')$ is equipped with the injective model structure. We denote by $\pi^\bullet : \text{ct}_{\mathcal{I}}(R') \rightarrow \Phi''$ the composition $\text{ct}_{\mathcal{I}}(R') \rightarrow \Phi'$ with $\Phi' \rightarrow \Phi''$. We denote by Φ''_n the image of $n \in \mathcal{I}$ by Φ'' .

$$\begin{array}{ccc} R & \xleftarrow{\sim} & R' \\ \tilde{\omega}^n \downarrow & & \downarrow \varpi^n \\ R/\mathfrak{a}^n & \xleftarrow{\sim} & \Phi'_n \\ & & \downarrow \lambda \\ & & \Phi''_n \end{array} \quad \begin{array}{c} \curvearrowright \\ \pi^n \end{array}$$

We set \mathcal{U} to be $(\langle \Phi''_1 \rangle_{R'}^{\mathcal{C}})^{\circ}$. We denote by $\iota : \mathcal{I} \rightarrow \mathcal{U}$ the functor which sends n to $\pi^n : R' \rightarrow \Phi''_n$. We denote by $\tilde{\iota} : \mathcal{I} \rightarrow \langle \Phi''_1 \rangle_{R'}^{\mathcal{C}}$ the functor which sends n to $\varpi^n : R' \rightarrow \Phi'_n$. By abuse of notations we denote by Γ the co-domain functors $\mathcal{U} \rightarrow \mathcal{C}(R')$ and $\langle \Phi'_1 \rangle_{R'}^{\mathcal{C}}$. We denote by $\kappa : \text{ct}(R') \rightarrow \Gamma$ the canonical morphism. Then we have isomorphisms $\Gamma \circ \tilde{\iota} \cong \Phi'$ and $\Gamma \circ \iota \cong \Phi''$. We can check that

under these isomorphisms we have $\kappa_{\tilde{\iota}} \cong \varpi^\bullet$ and $\kappa_{\iota} \cong \pi^\bullet$.

(6-12)

$$\begin{array}{ccc}
 \text{ct}_{\mathcal{I}}(R') & \xlongequal{\quad} & \text{ct}_{\mathcal{I}}(R') \\
 \downarrow \varpi^\bullet & & \downarrow \kappa_{\tilde{\iota}} \\
 \Phi' & \xlongequal{\quad} & \Gamma \circ \tilde{\iota} \\
 \downarrow & & \downarrow \\
 \Phi'' & \xlongequal{\quad} & \Gamma \circ \iota
 \end{array}
 \begin{array}{c}
 \nearrow \pi^\bullet \\
 \searrow \kappa_{\iota}
 \end{array}$$

We construct a bi-dual S functor over Φ''_1 by using Construction 3.1. Since there exists a injective cofibration $\pi^1 : R' \rightarrow \Phi''_1$, we may assume that the evaluation map $\epsilon_{R'} : R' \rightarrow S(R')$ is an injective cofibration. Hence by Theorem 3.4 the canonical morphism $S(\kappa) : \text{ct}_{\mathcal{U}} S(R') \rightarrow S \circ \Gamma$ exhibits $S(R')$ as a homotopy limit of R' .

We have the commutative diagram

(6-13)

$$\begin{array}{ccc}
 \text{ct}_{\mathcal{I}}(R') & \longrightarrow & \text{ct}_{\mathcal{I}}(S(R')) \\
 \downarrow \kappa_{\tilde{\iota}} & & \downarrow S(\kappa_{\tilde{\iota}}) \\
 \Gamma \circ \tilde{\iota} & \longrightarrow & S \circ \Gamma \circ \tilde{\iota} \\
 \downarrow & & \downarrow \\
 \Gamma \circ \iota & \longrightarrow & S \circ \Gamma \circ \iota
 \end{array}
 \begin{array}{c}
 \nearrow \kappa_{\iota} \\
 \searrow S(\kappa_{\iota})
 \end{array}$$

where the horizontal morphisms are induced by the unit map $\epsilon : 1 \rightarrow S$. The key of the proof is the following proposition.

Proposition 6.6. *The functor $\iota : \mathcal{I} \rightarrow \mathcal{U}$ is homotopy left cofinal.*

Before giving a proof of the proposition, we complete the proof of Theorem 6.1 with assuming Proposition 6.6.

We take the limits of the above diagram (6-13)

$$\begin{array}{ccc}
 & & S(R') \\
 & & \vdots \\
 \lim \Gamma \circ \tilde{\iota} & \dashrightarrow & \lim S \circ \Gamma \circ \tilde{\iota} \\
 \downarrow \simeq^{(2)} & & \downarrow \simeq^{(2)} \\
 \lim \Gamma \circ \iota & \xrightarrow{\simeq^{(2)}} & \lim S \circ \Gamma \circ \iota
 \end{array}
 \begin{array}{c}
 \nearrow \simeq^{(1)} \\
 \searrow
 \end{array}$$

By Proposition 6.6 we check the quasi-isomorphism (1) in the above diagram. By construction, for $n \geq 1$ the morphisms $\Gamma \circ \iota(\mathbf{geq}^{n+1,n}) = \Phi''(\mathbf{geq}^{n+1,n})$ and $\Gamma \circ \tilde{\iota}(\mathbf{geq}^{n+1,n}) = \Phi'(\mathbf{geq}^{n+1,n})$ is projective fibration. By Lemma 3.6.(3) the same things hold for $S \circ \Gamma \circ \iota$ and $S \circ \Gamma \circ \tilde{\iota}$. Therefore by Lemma 2.18 limits of those functors compute homotopy limits. Since homotopy limit is unique up to weak equivalence (Lemma 2.15), we check the quasi-isomorphisms (2) in the above diagram. Consequently we see that the dotted arrows in the above diagram are quasi isomorphisms. Therefore the following morphisms are quasi-isomorphisms of dg R' -modules

(6-14)

$$\lim_{n \rightarrow \infty} \Phi'_n \xrightarrow{\epsilon \sim} \lim_{n \rightarrow \infty} S(\Phi'_n) \xleftarrow{S(\kappa) \sim} S(R').$$

Note that since the quasi-isomorphisms $\Phi'_n \xrightarrow{\sim} R/\mathfrak{a}^n$ are taken as morphisms of dg-algebras, therefore the morphism $\lim_{n \rightarrow \infty} \Phi'_n \rightarrow \lim_{n \rightarrow \infty} R/\mathfrak{a}^n = \widehat{R}$ is also quasi-isomorphisms of dg-algebras. Note also that the middle term $\lim_{n \rightarrow \infty} S(\Phi'_n)$ in the above diagram has no obvious dg-algebra structure.

We denote by \mathcal{B} the bi-commutator $\mathcal{B}ic_{R'}(\Phi''_1) = S(R')$. Observe that the middle term $W := \lim_{n \rightarrow \infty} S(\Phi'_n)$ has a left $\widehat{R}' := \lim_{n \rightarrow \infty} \Phi'_n$ -module structure. Indeed the left multiplication $\lambda_r : \Phi'_n \rightarrow \Phi'_n$ of a element r of the dg-algebra Φ'_n is a homomorphism of R' -modules. Therefore $S(\Phi'_n)$ canonically has a structure of a left Φ'_n -module, which is compatible with the projective systems. Hence W has a left \widehat{R}' -module structure. On the other hands, $S(\Phi'_n)$ has a structure of a right \mathcal{B} -module structure which is compatible with the projective system. Hence W has a right \mathcal{B} -module structure.

Taking cohomology groups of diagram (6-14), we obtain isomorphism of graded modules

$$\widehat{R} \xrightarrow{\alpha \cong} H^\bullet(W) \xleftarrow{\cong \beta} H^\bullet(\mathcal{B}).$$

We can easily check that these morphisms satisfy the condition of the following lemma. Hence the map $\beta^{-1} \circ \alpha : \widehat{R} \rightarrow H^\bullet(\mathcal{B})$ is an algebra isomorphism. Moreover it is easy to see that this is compatible with the canonical homomorphisms $\mathbf{comp} : R \rightarrow \widehat{R}$ and $H^\bullet(\iota) : R \rightarrow H^\bullet(\mathcal{B})$.

Lemma 6.7. *Let A and B be rings and U a $A^{\text{op}} \otimes B$ -module. (i.e. a abelian group which has both left A -module structure and a right B -module structure such that $(am)b = a(mb)$ for $a \in A$, $m \in M$ and $b \in B$.) Assume that a left A -isomorphism $\alpha : A \rightarrow U$ and a right B -isomorphism $\beta : B \rightarrow U$ are given. If $\alpha(1_A) = \beta(1_B)$, then the map $\beta^{-1} \circ \alpha : A \rightarrow B$ is an algebra isomorphism.*

A proof is a straightforward calculation and left to the readers.

Let $\tau^{\leq 0} : \text{dgCat} \rightarrow \text{dgCat}$ be the smart truncation functor. Then we have canonical natural transformations $\nu : \tau^{\leq 0} \rightarrow 1_{\text{dgCat}}$ and $\nu' : \tau^{\leq 0} \rightarrow H^0(-)$. Observe that ν and ν' are quasi-isomorphisms for dg-categories such that the cohomology groups of all Hom-complex are concentrated in degree 0. By the above isomorphism we see that the cohomology of $\mathbb{B}ic_R(R/\mathfrak{a}) \simeq \mathcal{B}$ is concentrated in degree 0. Since the dg-algebra R' is quasi-isomorphic to the ring R , the cohomology group of R' is also concentrated in degree 0. Therefore by the above natural transformations the canonical morphism $\iota : R' \rightarrow \mathbb{B}ic_R(R/\mathfrak{a})$ corresponds to $H^0(\iota) : R \rightarrow H^0(\mathbb{B}ic_R(R/\mathfrak{a}))$. Hence we obtain the following desired commutative diagram in the homotopy category $\text{Ho}(\text{dgCat})$

$$\begin{array}{ccc} & R & \\ \text{comp} \swarrow & \downarrow H^0(\iota) & \searrow \iota \\ \widehat{R} & \xrightarrow{\cong} H^0(\mathbb{B}ic_R(R/\mathfrak{a})) & \xrightarrow{\cong} \mathbb{B}ic_R(R/\mathfrak{a}). \end{array}$$

In the rest of this section we devote to prove Proposition 6.6. Namely we prove that the ∞ -functor $N(\iota) : N(\mathcal{I}) \rightarrow N(\mathcal{U})$ is left cofinal.

We denote by V the full sub ∞ category of $N(\mathcal{U})$ consisting of π^n . Then we have $N(\iota) = \xi \circ \eta$ where $\eta : N(\mathcal{I}) \rightarrow V$ and $\xi : V \rightarrow N(\mathcal{U})$ be the canonical inclusions. By [20, Proposition 4.1.1.3] a composition of left cofinal functors is left cofinal. Therefore it is enough to prove that ξ and η are left cofinal. We prove these in Lemma 6.8 and Lemma 6.9.

Lemma 6.8. *The functor η is left cofinal.*

Proof. By Theorem 2.19 it is enough to show that η_n is weakly contractible for each $n \geq 1$. To prove this, it is enough to show that any simplicial map $f : X \rightarrow \eta_n$ from a finite simplicial set X factors through $*$ in the homotopy category $\text{Ho}(\text{sSet})$.

Recall that $\eta_n = N(\mathcal{I}) \times_V V_n$. We set $f_1 = \text{pr}_1 \circ f$, $f_2 = \text{pr}_2 \circ f$ where pr_1, pr_2 are the first and second projection. Since $\mathcal{I} \cong (\mathbb{Z}_{\geq 1})^{\text{op}}$, the simplicial map $f_1 : X \rightarrow N(\mathcal{I})$ is homotopic to some constant map. More precisely, there exists a simplicial map $H' : X \times \Delta^1 \rightarrow N(\mathcal{I})$ such that the

restriction $H'|_{X \times \{1\}}$ to $X \times \{1\} \cong X$ is f_1 and the restriction $H'|_{X \times \{0\}}$ to $X \times \{0\}$ is the constant functor with the value $m := \max\{f_1(x) \mid x \in X\}$.

Since the simplicial map $\{1\} \rightarrow \Delta^1$ is isomorphic to the horn inclusion $\Lambda_1^1 \rightarrow \Delta^1$, by [20, Corollary 2.1.2.7] the simplicial map $X \times \{1\} \rightarrow X \times \Delta^1$ is right anodyne. Since by [20] the co-domain functor $V_{/n} \rightarrow V$ is a right fibration, there exists a simplicial map $H'' : X \times \Delta^1 \rightarrow V_{/n}$ which complete the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f_2} & V_{/n} \\ \downarrow & \nearrow H'' & \downarrow \\ X \times \Delta^1 & \xrightarrow{H'} & N(\mathcal{I}) \xrightarrow{\eta} V \end{array}$$

Then the functor H', H'' gives the functor $H : X \times \Delta^1 \rightarrow \eta_{/n}$ such that the restriction $H|_{X \times \{1\}}$ is f . We denote by g the restriction $H|_{X \times \{0\}}$. The map H gives a homotopy between f and g . Therefor it is enough to prove that g factors through $*$ in the homotopy category $\text{Ho}(\text{sSet})$.

Since $g \circ \text{pr}_1 = H'|_{X \times \{0\}}$ is the constant map with the value m , the map g factors through $\text{Hom}_V^R(m, n)$, (Recall that $\text{Hom}_V^R(m, n) = \{m\} \times_V V_{/n}$)

$$\begin{array}{ccccc} X & & & & \\ & \searrow g & & & \\ & & \text{Hom}_V^R(m, n) & \xrightarrow{\quad} & \eta_{/n} \xrightarrow{\text{pr}_2} V_{/n} \\ & \swarrow & \downarrow & \downarrow \text{pr}_1 & \downarrow \\ & & \{m\} & \xrightarrow{\quad} & N(\mathcal{I}) \xrightarrow{\eta} V \end{array}$$

By [20, 2.2], the complex $\text{Hom}_V^R(m, n)$ is weak homotopy equivalent to $\text{Map}_{\mathcal{U}}(\pi^m, \pi^n)$. We finishes the proof by showing that $\text{Map}_{\mathcal{U}}(\pi^m, \pi^n)$ is weakly contractible. We check that $\pi_i(\text{Map}_{\mathcal{U}}(\pi^m, \pi^n)) = 0$ for $i \geq 0$. Recall that the mapping complex $\text{Map}_{\mathcal{U}}(\pi^m, \pi^n)$ fits into the pull-back diagram

$$\begin{array}{ccc} \text{Map}_{\mathcal{U}}(\pi^m, \pi^n) & \longrightarrow & \text{Map}_{R'}(\Phi''_m, \Phi''_n) \\ \downarrow & & \downarrow \text{ind} \\ \{\pi^n\} & \longrightarrow & \text{Map}_{R'}(R', \Phi''_n) \end{array}$$

where **ind** is the induced morphism $\text{ind} := \text{Map}(\pi^m, \Phi''_n)$. Since π^m is a injective cofibration and Φ''_n is injectively fibrant, the map $\text{Map}(\pi^m, \Phi''_n)$ is a fibration of simplicial set. Therefore we the homotopy groups $\pi_i(\text{Map}_{\mathcal{U}}(\pi^m, \pi^n))$ fits into the homotopy long exact sequence.

$$\begin{aligned} \pi_{i+1}(\text{Map}_{R'}(\Phi''_m, \Phi''_n)) &\xrightarrow{\pi_{i+1}(\text{ind})} \pi_{i+1}(\text{Map}_{R'}(R', \Phi''_n)) \rightarrow \pi_i(\text{Map}_{\mathcal{U}}(\pi^m, \pi^n)) \\ &\rightarrow \pi_i(\text{Map}_{R'}(\Phi''_m, \Phi''_n)) \xrightarrow{\pi_i(\text{ind})} \pi_i(\text{Map}_{R'}(R', \Phi''_n)) \end{aligned}$$

The canonical exact sequence $0 \rightarrow \mathfrak{a}^m \rightarrow R \xrightarrow{\tilde{\omega}^m} R/\mathfrak{a}^m \rightarrow 0$ induces an exact triangle

$$\mathbb{R} \text{Hom}_R(\mathfrak{a}^m[1], R/\mathfrak{a}^n) \rightarrow \mathbb{R} \text{Hom}_R(R/\mathfrak{a}^m, R/\mathfrak{a}^n) \xrightarrow{\widetilde{\text{ind}}} \mathbb{R} \text{Hom}_R(R, R/\mathfrak{a}^n) \rightarrow .$$

where $\widetilde{\text{ind}}$ is the morphism induced from $\tilde{\omega}^n$. Since we have

$$\text{H}^{-i}(\mathbb{R} \text{Hom}_R(\mathfrak{a}^m[1], R/\mathfrak{a}^n)) \cong \text{Ext}^{-i-1}(\mathfrak{a}^m, R/\mathfrak{a}^n) = 0,$$

the morphism $H^0(\widetilde{\mathbf{ind}})$ is injective and the morphisms $H^i(\widetilde{\mathbf{ind}})$ are isomorphisms for $i \geq 1$. On the other hands, we have the commutative diagram for $i \geq 0$

$$\begin{array}{ccc} \pi_i(\mathrm{Map}_{R'}(\Phi''_m, \Phi''_n)) & \xlongequal{\sim} & H^{-i}(\mathbb{R} \mathrm{Hom}_R(R/\mathfrak{a}^m, R/\mathfrak{a}^n)) \\ \pi_i(\mathbf{ind}) \downarrow & & \downarrow H^{-i}(\widetilde{\mathbf{ind}}) \\ \pi_i(\mathrm{Map}_{R'}(R', \Phi''_n)) & \xlongequal{\sim} & H^{-i}(\mathbb{R} \mathrm{Hom}_R(R, R/\mathfrak{a}^n)) \end{array}$$

Hence from the above homotopy long exact sequence, we conclude that all the homotopy groups of $\mathrm{Map}_{\mathcal{U}}(\pi^m, \pi^n)$ vanish. This finishes the proof. \square

Lemma 6.9. *The functor ξ is left cofinal.*

Proof. We can prove that the ∞ -category $N(\mathcal{U})$ is co-filtered as in the same way of the proof of Lemma 4.4. In view of Lemma 2.20 it is enough to show that for any morphism $k : R' \rightarrow K$ in $\mathcal{C}(R')$ with $K \in (\langle \Phi''_1 \rangle^{\mathcal{C}})^{\circ}$ there exists a morphism $\pi^n \rightarrow k$ for some $n \geq 1$. By Lemma 2.1 it is enough to prove the same statement in the derived category $\mathcal{D}(R')$.

The quasi-isomorphism $R' \xrightarrow{\sim} R$ induces the equivalence $\mathcal{D}(R') \simeq \mathcal{D}(R)$. Therefore the problem is reduced to prove the same statement in $\mathcal{D}(R)$. Observe that under the equivalence, the object Φ''_1 is isomorphic to R/\mathfrak{a} and hence the thick subcategory $\langle \Phi''_1 \rangle^{\mathcal{D}}$ is equivalent to the thick subcategory $\langle R/\mathfrak{a} \rangle^{\mathcal{D}}$. By the assumption that $\mathcal{D}_{\mathfrak{a}\text{-tor}}(R) \simeq \mathcal{D}(\mathfrak{a}\text{-tor})$, an object K of $\langle R/\mathfrak{a} \rangle$ is quasi-isomorphic to a complex each terms of which is a \mathfrak{a} -torsion R module. Hence we immediately see that in the derived category $\mathcal{D}(R)$ the morphism $k : R \rightarrow K$ with $K \in \langle R/\mathfrak{a} \rangle^{\mathcal{D}}$ factors through the canonical projection $R \rightarrow R/\mathfrak{a}^n$ for some n . By Lemma 2.1 we obtain a morphism $\pi^n \rightarrow k$ in \mathcal{U} . \square

Now we finish the proof of Theorem 6.1.

7 Smashing localization via derived bi-commutator

7.1 Smashing localization of dg-categories

A functor $F : \mathcal{T} \rightarrow \mathcal{S}$ between triangulated categories is called a *smashing localization* if F has a fully faithful right adjoint $G : \mathcal{S} \hookrightarrow \mathcal{T}$ which preserves direct sums. A dg-functor $f : \mathcal{A} \rightarrow \mathcal{B}$ of small dg-categories is called *smashing localization (or homological epimorphisms)* if the restriction functor $f_* : \mathcal{D}(\mathcal{B}) \rightarrow \mathcal{D}(\mathcal{A})$ is fully faithful. In the study of smashing localization, we mainly interest in the functors $\mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{B})$ between derived categories. (See e.g. [15, 25].) Therefore we consider smashing localizations $f : \mathcal{A} \rightarrow \mathcal{B}$ of a dg-category \mathcal{A} up to Morita equivalence of \mathcal{B} .

Let $f : \mathcal{A} \rightarrow \mathcal{B}'$ be a smashing localization. We decompose f into a cofibration $g : \mathcal{A} \rightarrow \mathcal{B}$ followed by a trivial fibration $h : \mathcal{B} \xrightarrow{\sim} \mathcal{B}'$. Then f and g induce the same smashing localization of the derived category $\mathcal{D}(\mathcal{A})$. Hence in the study of smashing localization, we may assume that $f : \mathcal{A} \rightarrow \mathcal{B}$ is a cofibration of dg-categories. Moreover by the following lemma, we may replace f with the essential image of a cofibration.

Lemma 7.1. *Let $f : \mathcal{A} \rightarrow \mathcal{B}'$ be a smashing localization. We decompose f into an essentially surjection $g : \mathcal{A} \rightarrow \mathcal{B}$ followed by a fully faithful dg-functor $h : \mathcal{B} \rightarrow \mathcal{B}'$. Then h induces Morita equivalence, i.e., $h_* : \mathcal{D}(\mathcal{B}') \rightarrow \mathcal{D}(\mathcal{B})$ gives an equivalence, and hence g is also smashing localization.*

Proof. First we show that h_* is fully faithful. Since f_* is fully faithful, the counit morphism $f^* f_* N \rightarrow N$ is an isomorphism for any $N \in \mathcal{D}(\mathcal{B}')$. In particular every object of $\mathcal{D}(\mathcal{B}')$ is of form $f^* K$ for some $K \in \mathcal{D}(\mathcal{A})$. Therefore it is enough to prove that the map $\mathrm{Hom}_{\mathcal{D}(\mathcal{B}')} (f^* K, f^* L) \rightarrow$

$\text{Hom}_{\mathcal{D}(\mathcal{B})}(h_*f^*K, h_*f^*L), \psi \mapsto h_*(\psi)$ is an isomorphism for $K, L \in \mathcal{D}(\mathcal{A})$. However it is easily shown by using the equation $f^* = h^*g^*$ and the natural isomorphism $h_*h^*h^* \cong h^*$.

Since $h : \mathcal{B} \rightarrow \mathcal{B}'$ is fully faithful, the unite map $b^\wedge \rightarrow h_*h^*(b^\wedge)$ is an isomorphism. Recall that we have a canonical isomorphism $g^*(a^\wedge) \cong (ga)^\wedge$. Therefore we have $h_*f^*(a^\wedge) \cong h_*h^*g^*(a^\wedge) \cong g^*(a^\wedge)$. Since g is essentially surjective, the set $\{g^*(a^\wedge) \mid a \in \mathcal{A}\}$ of modules equals to the set $\{b^\wedge \mid b \in \mathcal{B}\}$. Therefore the modules $h_*(f^*(a^\wedge))$ form a set of compact generators of $\mathcal{D}(\mathcal{B})$. On the other hands by [25, Lemma in page 1231] the modules $(fa)^\wedge \cong f^*(a^\wedge)$ form a set of compact generators of $\mathcal{D}(\mathcal{B}')$. Since the functor h_* is exact and commutes with arbitrary direct sum, we conclude that h_* is essentially surjective. \square

A notion of pure monomorphism and pure injective object for compactly generated triangulated category were introduced by Krause [15] to study smashing localizations.

Definition 7.2. (1) A morphism $M \rightarrow N$ in $\mathcal{D}(\mathcal{A})$ is said to be a *pure monomorphism* if the induced map $\text{Hom}_{\mathcal{D}(\mathcal{A})}(P, M) \rightarrow \text{Hom}_{\mathcal{D}(\mathcal{A})}(P, N)$ is a monomorphism for all $P \in \text{Perf}\mathcal{A}$.

(2) A morphism $M \rightarrow N$ in $\mathcal{D}(\mathcal{A})$ is said to be a *cohomologically monomorphism* if the induced map $H^*(M) \rightarrow H^*(N)$ is a monomorphism.

(3) An object J of $\mathcal{D}(\mathcal{A})$ is called *pure injective* (resp. *cohomologically injective* (C.I. for short.)) if $\psi : M \rightarrow N$ is pure monomorphism (resp. cohomologically monomorphism), then every morphism $M \rightarrow J$ factors through ψ .

We call an object E of $\mathcal{D}(\mathcal{A})$ a *co-generator* if, for $M \in \mathcal{D}(\mathcal{A})$, the condition $\text{Hom}_{\mathcal{D}(\mathcal{A})}(M, E) = 0$ implies $M = 0$. (We remark that this definition is not a direct opposite version of generators for triangulated categories. See e.g. [30, 3.1.1.].) We call an object J of $\mathcal{C}(\mathcal{A})$ a pure injective, a C.I., a co-generator if it is so as an object of $\mathcal{D}(\mathcal{A})$.

We can easily prove the following lemma.

Lemma 7.3. (1) A pure injective mono morphism is a cohomologically monomorphism.

(2) A C.I. object is a pure injective object.

Example 7.4. Let R be a ring and J be an injective R -module. Then J is a C.I. object. Moreover if J is an injective co-generator of $\text{Mod } R$, then $\prod_{n \in \mathbb{Z}} J[n]$ is a C.I. co-generator of $\mathcal{D}(R)$.

Note that by [15, Proposition 2.6] if $f : \mathcal{A} \rightarrow \mathcal{B}$ is a smashing localization of dg-categories then the restriction functor $f_* : \mathcal{D}(\mathcal{B}) \rightarrow \mathcal{D}(\mathcal{A})$ sends pure-injective objects to pure-injective object. The following is main theorem of this section, which says that every smashing localization is obtained as a derived bi-commutator of some pure-injective object.

Theorem 7.5. Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be an essentially surjective smashing localization such that the induced adjoint pair

$$f^* : \mathcal{C}(\mathcal{A}) \rightleftarrows \mathcal{C}(\mathcal{B}) : f_*$$

is a Quillen pair where both $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{B})$ are equipped with the injective model structures. Let J be a pure injective co-generator of $\mathcal{D}(\mathcal{B})$. Then we have quasi-equivalence of dg-categories over \mathcal{A}

$$\mathcal{B} \simeq \mathbb{B}\text{ic}_{\mathcal{A}}(f_*J')$$

where J' is a large enough product J^{Π_κ} of J .

Remark 7.6. Nicolás and Saorin [25] proved that for any smashing localization $F : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{S}$ there exists a subcategory $\mathcal{I} \subset \mathcal{D}(\mathcal{A})$ such that the functor $\mathbb{L}\iota_{\mathcal{I}}^* : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(\mathbb{B}\text{ic}_{\mathcal{A}}(\mathcal{I}))$ induced from the canonical morphism $\iota_{\mathcal{I}} : \mathcal{A} \rightarrow \mathbb{B}\text{ic}_{\mathcal{A}}(\mathcal{I})$ is equivalent to F .

In a proof of Theorem 7.5 an essential part is the following theorem.

Theorem 7.7. *Let J be an injective fibrant cohomologically injective co-generator. We fix a set \mathcal{M} of objects of $\mathcal{C}(\mathcal{A})$. If we take a product $J' := J^{\Pi\kappa}$ of copies of J over a large enough cardinal κ , then for each M in \mathcal{M} , the bi-dual $M^{\otimes\otimes}$ of M taken over J' is isomorphic to M .*

More precisely, if we choose an injectively fibrant representative of J' appropriately, then the evaluation map $\epsilon_M : M \rightarrow S(M)$ is a quasi-isomorphism.

A proof of Theorem 7.5 with assuming Theorem 7.7. By Theorem 7.7, there exists an injective fibrant representative of a fixed injective co-generator $J \in \mathcal{D}(\mathcal{B})$, which will be also denoted by $J' \in \mathcal{C}(\mathcal{B})$, such that if we take a large enough cardinal κ , the the product $J' := J^{\Pi\kappa}$ satisfies the following property: let $S_{J'}$ be the bi-dual taken over J' , for any $b \in \mathcal{B}$ the evaluation map $\epsilon_{b^\wedge} : b^\wedge \rightarrow S_{J'}(b^\wedge)$ is a quasi-isomorphism. Then by Lemma 3.3 the canonical functor $\iota_{J'} : \mathcal{B} \rightarrow \mathbb{B}\mathrm{ic}_{\mathcal{B}}(J')$ is quasi-equivalence. By Proposition 5.2 we have the commutative diagram in $\mathrm{Ho}(\mathrm{dgCat})$

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ & \searrow \iota_J & \parallel \\ & & \mathbb{B}\mathrm{ic}_{\mathcal{A}}(f_* J'). \end{array}$$

Now we finish the proof. □

7.2 A proof of Theorem 7.7.

We denote by $H^\bullet(\mathcal{A})$ the homotopy category of the dg-category \mathcal{A} . Namely this is a graded category such that the objects of $H^\bullet(\mathcal{A})$ is the same with that of \mathcal{A} and the Hom-graded module is given by $H^\bullet(\mathcal{A})(a, b) := H^\bullet(\mathcal{A}^\bullet(a, b))$.

By a version of [15, Theorem 1.8. Corollary 1.9] (see also [7]), we can easily deduce the following lemma.

Lemma 7.8. (1) *Let M be an object of $\mathcal{D}(\mathcal{A})$ and J a cohomologically injective object of $\mathcal{D}(\mathcal{A})$. Then the map $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(M, J) \rightarrow \mathrm{Hom}_{\mathcal{G}(H^\bullet(\mathcal{A}))}(H^\bullet(M), H^\bullet(J))$ induced from the functor $H^\bullet : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{G}(H^\bullet(\mathcal{A}))$ is an isomorphism.*

(2) *An object $J \in \mathcal{D}(\mathcal{A})$ is a C.I. object (resp. injective co-generator) if and only if the cohomology group $H^\bullet(J)$ is a injective object (resp. injective co-generator) of $\mathcal{G}(H^\bullet(\mathcal{A}))$.*

(3) *Let J be a C.I. co-generator of $\mathcal{D}(\mathcal{A})$. Then for any object $M \in \mathcal{D}(\mathcal{A})$ there exists a C.I. morphism $M \rightarrow J^{\Pi\kappa}$ for some cardinal κ .*

(4) *Let J be a C.I. co-generator of $\mathcal{D}(\mathcal{A})$. The graded $H^\bullet(\mathcal{A})$ -module $H^\bullet(M)$ admits a resolution*

$$(7-15) \quad 0 \rightarrow H^\bullet(M) \rightarrow H^\bullet(J^{\Pi\kappa_0^M}) \rightarrow H^\bullet(J^{\Pi\kappa_1^M}) \rightarrow H^\bullet(J^{\Pi\kappa_2^M}) \rightarrow \dots$$

for some cardinal κ_i^M for $i \geq 0$.

Let \mathcal{M} be a set of objects of $\mathcal{D}(\mathcal{A})$. The following theorem tells us that if we set $\kappa := \sup\{\kappa_i^M \mid M \in \mathcal{M}, i \geq 0\}$ where κ_i^M is a cardinal appearing in the above resolution (7-15), then the bi-duality $S(M)$ taken over $J^{\Pi\kappa}$ is quasi-isomorphic to M .

Theorem 7.9. *Let J be a cohomological injective object of $\mathcal{D}(\mathcal{A})$. We fix a set \mathcal{M} of objects of $\mathcal{C}(\mathcal{A})$. Assume that for each $M \in \mathcal{M}$ there exist objects J_M^i for $i \geq 0$ such that the graded $H^\bullet(\mathcal{A})$ -module $H^\bullet(M)$ admits a resolution*

$$(7-16) \quad 0 \rightarrow H^\bullet(M) \xrightarrow{\tau_M} H^\bullet(J_M^0) \xrightarrow{d_M^0} H^\bullet(J_M^1) \xrightarrow{d_M^1} H^\bullet(J_M^2) \xrightarrow{d_M^2} \dots$$

whose each term J_M^i is a direct summand of a finite direct sum of J_M . Then the evaluation map $\epsilon_M : M \rightarrow S_J(M)$ is a quasi-isomorphism.

More precisely, if we choose an injectively fibrant representative of J appropriately, then the evaluation map is a quasi-isomorphism

$$\epsilon_M : M \rightarrow S(M)$$

By Example 7.4 we obtain the following corollary.

Corollary 7.10. *Let R be a ring and J an injective R -module. Assume that an R -module M admits an injective resolution*

$$0 \rightarrow M \rightarrow J^0 \rightarrow J^1 \rightarrow J^2 \rightarrow \dots$$

such that each term J^i is a direct summand of finite direct sum of J . Then the evaluation map $\epsilon_M : M \rightarrow M^{\otimes \otimes}$ is a quasi-isomorphism.

Remark 7.11. This corollary is already obtained by Shamir [32] in a different way.

In the rest of this section we devote to prove Theorem 7.9.

We proceed a proof of Theorem 7.9. Since the derived bi-duality functor $S(-)$ preserves quasi-isomorphisms, we may assume that all objects M of \mathcal{M} are injectively fibrant.

Using the following Lemma 7.12 we inductively construct a $\#$ -exact sequence of dg \mathcal{A} -modules for each $M \in \mathcal{M}$

$$(7-17) \quad 0 \rightarrow M \xrightarrow{\delta_M^{-1}} \tilde{J}_M^0 \xrightarrow{\delta_M^0} \tilde{J}_M^1 \xrightarrow{\delta_M^1} \tilde{J}_M^2 \xrightarrow{\delta_M^2} \dots$$

such that

1. cohomology $H^\bullet(-)$ coincide with the given resolution (7-16), (hence \tilde{J}_M^i is an injectively fibrant representative of J_M^i),
2. each $\text{coker}(\delta_M^i)$ is injectively fibrant for $i \geq -1$.

Lemma 7.12. *Let \tilde{N} be an injectively fibrant object of $\mathcal{C}(\mathcal{A})$. Assume that a morphism $r : H^\bullet(\tilde{N}) \rightarrow H^\bullet(L)$ is given with some C.I. object L of $\mathcal{D}(\mathcal{A})$. Then there exists an injectively fibrant replacement \tilde{L} of L and a morphism $\underline{r} : \tilde{N} \rightarrow \tilde{L}$ such that $H^\bullet(\underline{r}) = r$ and $\text{coker}(\underline{r})$ is injectively fibrant.*

Proof. By Lemma 7.8 there exists a morphism $r' : \tilde{N} \rightarrow L$ in $\mathcal{D}(\mathcal{A})$ such that $H^\bullet(r') = r$. We pick an injectively fibrant replacement \tilde{L}' and a representative $\underline{r}' : \tilde{N} \rightarrow \tilde{L}'$ of r' . Let $C := c(1_{\tilde{N}})$ be the cone of the identity $1_{\tilde{N}}$ and $\iota := {}^t(1_{\tilde{N}}, 0) : \tilde{N} \rightarrow C$ the canonical inclusion. We set $\tilde{L} := \tilde{L}' \oplus C$ and $\underline{r} := {}^t(\underline{r}', \iota) : \tilde{N} \rightarrow \tilde{L}$. Then we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{N} & \xrightarrow{\underline{r}} & \tilde{L} & \longrightarrow & \text{coker}(\underline{r}) \longrightarrow 0 \\ & & \parallel & & \downarrow (0, 1_C) & & \downarrow \\ 0 & \longrightarrow & \tilde{N} & \xrightarrow{\iota} & C & \longrightarrow & \tilde{N}[1] \longrightarrow 0 \end{array}$$

where the top and the bottom rows are $\#$ -exact. By snake Lemma we obtain the $\#$ -exact sequence $0 \rightarrow \tilde{L}' \rightarrow \text{coker}(\underline{r}) \rightarrow \tilde{N}[1] \rightarrow 0$. Therefore by Lemma 2.6 we conclude that $\text{coker}(\underline{r})$ is injectively fibrant. Since the cone C is weakly contractible, we see that $H^\bullet(\underline{r}) = r$. \square

We choose an injectively fibrant representative \tilde{J} of J . Then for $M \in \mathcal{M}$ and $i \geq 0$, the module \tilde{J}_M^i belongs to $(\langle \tilde{J} \rangle^c)^\circ$.

The above resolution is a main tool for the proof. We need to fix notations. From now for simplicity we denote δ_M^i , \tilde{J}_M^i and \tilde{J}_M by δ_M^i , J^i and J respectively. Since we prove that the evaluation map ϵ_M is a quasi-isomorphism for each M separately, these modifications are harmless for our purpose. We set $M^0 := M$ and $M^n = \text{coker}(\delta^{n-2})$ for $n \geq 1$. We denote by $\lambda^n : M^n \hookrightarrow J^n$ and $\rho : J^n \rightarrow M^{n+1}$ the canonical morphisms. Therefore we have $\delta^n := \lambda^{n+1} \circ \rho^n$.

$$\begin{array}{ccccc} \xrightarrow{\delta^{n-1}} J^n & \xrightarrow{\delta^n} & J^{n+1} & \xrightarrow{\delta^{n+1}} & \\ & \searrow \rho^n & \swarrow \lambda^{n+1} & & \\ & & M^{n+1} & & \end{array}$$

For $n \leq m$ we denote by $I^{[n,m]}$ the totalization of the complex

$$J^n \xrightarrow{\delta^n} J^{n+1} \xrightarrow{\delta^{n+1}} \dots \xrightarrow{\delta^{m-1}} J^m.$$

More precisely, the dg \mathcal{A} -module $I^{[n,m]}$ is defined in the following way: the underlying graded module of $I^{[n,m]}$ is given by

$$J^m[-(m-n)] \bigoplus J^{m-1}[-(m-n-1)] \bigoplus \dots \bigoplus J^{n+1}[-1] \bigoplus J^n$$

and the differential $d_{I^{[n,m]}}$ is given by

$$(-1)^n \begin{pmatrix} (-1)^{m-n} d_{J^m} & \delta^{m-1}[-(m-n-1)] & 0 & \dots & 0 & 0 \\ 0 & (-1)^{m-n-1} d_{J^{m-1}} & \delta^{m-2}[-(m-n-2)] & \dots & 0 & 0 \\ 0 & 0 & (-1)^{m-n-2} d_{J^{m-2}} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -d_{J^{n+1}} & \delta^n \\ 0 & 0 & 0 & \dots & 0 & d_{J^n} \end{pmatrix}$$

We denote by π_n^m the morphism ${}^t(0, 0, \dots, 0, \lambda^n) : M^n \rightarrow I^{[n,m]}$. For $\ell > m > n$ we denote by $\varphi_n^{\ell,m}$ the morphism $I^{[n,\ell]} \rightarrow I^{[n,m]}$ induced from the morphism of complexes

$$\begin{array}{ccccccc} J^n & \longrightarrow & J^{n+1} & \longrightarrow & \dots & \longrightarrow & J^m & \longrightarrow & J^{m+1} & \longrightarrow & \dots & \longrightarrow & J^\ell \\ \downarrow 1_{J^n} & & \downarrow 1_{J^{n+1}} & & & & \downarrow 1_{J^m} & & \downarrow 0 & & & & \downarrow 0 \\ J^n & \longrightarrow & J^{n+1} & \longrightarrow & \dots & \longrightarrow & J^m & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 \end{array}$$

Then we have $\pi_n^m = \varphi_n^{\ell,m} \circ \pi_n^\ell$. We set $I^n := I^{[0,n]}$, $\pi^n := \pi_0^n$ and $\varphi^{m,n} := \varphi_0^{m,n}$.

$$\begin{array}{ccccccc} & & M & & & & \\ & & \downarrow \pi^n & \searrow \pi^{n-1} & & & \\ \dots & \xrightarrow{\varphi^{n+1,n}} & I^n & \xrightarrow{\varphi^{n,n-1}} & I^{n-1} & \xrightarrow{\varphi^{n-1,n-2}} & \dots \end{array}$$

The limit $\lim_{n \rightarrow \infty} I^n$ is the totalization of $J^0 \rightarrow J^1 \rightarrow \dots$. By [12] the system $\{\pi^n\}_{n \geq 0}$ of morphisms induces a quasi-isomorphism

$$\pi^\infty : M \xrightarrow{\sim} \lim_{n \rightarrow \infty} I^n.$$

We denote by \mathcal{U} the full subcategory $(\langle J \rangle_{M/}^c)^\circ$. We denote by \mathcal{I} the category $(\mathbb{Z}_{\geq 0})^{\text{op}}$. We define a functor $\Phi : \mathcal{I} \rightarrow \mathcal{U}$ to be a functor which sends an object n to π^n and a unique morphism $\mathbf{geq}^{m,n} : m \rightarrow n$ to $\varphi^{m,n}$.

The key of the proof of Theorem 7.9 is the following proposition.

Proposition 7.13. *The functor Φ is homotopy left cofinal.*

First we assume Proposition 7.13 and prove Theorem 7.9. We have the commutative diagram

$$(7-18) \quad \begin{array}{ccc} \text{ct}_{\mathcal{I}} M & \xrightarrow{\epsilon_M} & \text{ct}_{\mathcal{I}} S(M) \\ \kappa \downarrow & & \downarrow S(\kappa) \\ \Gamma|_{\mathcal{I}} & \xrightarrow{\epsilon_{\Gamma}} & S \circ \Gamma|_{\mathcal{I}}. \end{array}$$

The bottom arrow is a weak equivalence. By Proposition 7.13 the right vertical arrow exhibits $S(M)$ as a homotopy limit of $S \circ \Gamma$. We claim that the left vertical arrow exhibits M as a homotopy limit of $\Gamma|_{\mathcal{I}}$. Indeed we see that there exists a canonical isomorphism $\lim_{\mathcal{I}} \Gamma|_{\mathcal{I}} \cong \lim_{n \rightarrow \infty} I^n$ and under this isomorphism the morphism $\lim \kappa : M \rightarrow \lim_{\mathcal{I}} \Gamma|_{\mathcal{I}}$ corresponds to the quasi-isomorphism $\pi^\infty : M \xrightarrow{\sim} \lim_{n \rightarrow \infty} I^n$. Since each $\Gamma(\mathbf{geq}^{n,n-1}) = \varphi^{n,n-1}$ is a projective fibration, the limit $\lim \Gamma|_{\mathcal{I}}$ computes a homotopy limit of $\Gamma|_{\mathcal{I}}$. Therefore we verify the claim.

Taking limits of the above diagram (7-18), by the uniqueness of homotopy limit up to weak equivalence (Lemma 2.15), we conclude that the evaluation map $\epsilon_M : M \rightarrow S(M)$ is a quasi-isomorphism. This finishes the proof of Theorem 7.9 except Proposition 7.13.

In the rest of this section we devote to prove Proposition 7.13. According to Theorem 2.19, it is enough to verify that the simplicial set $N(\Phi)_{/k}$ is non-empty and weakly contractible for each object $k : M \rightarrow K$ of \mathcal{U} . First we prove

Lemma 7.14. *The simplicial set $N(\Phi)_{/k}$ is non-empty. Namely for some $m \geq 0$ there exists a morphism $\psi : I^m \rightarrow K$ which satisfies $k = \psi \circ \pi^m$.*

To prove this lemma and complete the proof of Proposition 7.13, we need little more informations of the resolution (7-17). We denote by ξ_n^m the canonical projection $I^{[n,m]} \rightarrow \text{coker}(\pi_n^m)$.

Lemma 7.15. *There exists a injective trivial fibration $\eta_n^m : \text{coker}(\pi_n^m) \xrightarrow{\sim} M^{m+1}[-(m-n)]$ such that the composite morphism $\eta_n^m \circ \xi_n^m$ is equal to $(\rho^m[-(m-n)], 0, \dots, 0)$.*

First we claim

Lemma 7.16. (1) *Assume that the following commutative diagram such that the top row is $\#$ -exact is given in $\mathcal{C}(\mathcal{A})$*

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow & Z \longrightarrow 0 \\ & & \downarrow \alpha & & \parallel & & \downarrow h \\ & & \text{cc}(g) & \xrightarrow{(0, 1_{Y'})} & Y' & \xrightarrow{g} & Z' \end{array}.$$

Then the morphism $\alpha := {}^t(0, f) : X \rightarrow Z'[-1] \oplus Y'$ of graded modules become the morphism $\alpha : X \rightarrow \text{cc}(g)$ in $\mathcal{C}(\mathcal{A})$ and completes the above diagram. Moreover we have a canonical isomorphism $\text{coker}(\alpha) \cong \text{c}(h)[-1]$.

(2) *Assume that the commutative diagram such that the vertical arrows are injective fibrations is given in $\mathcal{C}(\mathcal{A})$*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \wr & & \downarrow \wr \\ X' & \xrightarrow{f'} & Y'. \end{array}$$

Then the induced morphism $c(f) \rightarrow c(f')$ is an injective fibration.

(3) Let $f : X \rightarrow Y$ be a morphism of dg \mathcal{A} -modules. We denote by g the canonical projection $Y \rightarrow \text{coker}(f)$. Then we have the $\#$ -exact sequence of dg \mathcal{A} -modules

$$0 \rightarrow \ker(f) \rightarrow c(1_X) \xrightarrow{\text{diag}(f, 1_{X[1]})} c(f) \xrightarrow{(g, 0)} \text{coker}(f) \rightarrow 0$$

In particular, if f is an injective cofibration and X is injective fibrant, then the morphism $(g, 0)$ is a trivial injective fibration.

The proof is straightforward. The last statement of (3) follows from Lemma 2.5 and Lemma 2.6.

Proof of Lemma 7.15. We prove by induction on $m \geq n$. For the case $m = n$, it is enough to set $\eta_n^n := 1_{M^{n+1}}$, since $\text{coker}(\pi_n^n) = M^{n+1}$. We assume that the case when m is verified. Observe that the module $I^{[n, m+1]}$ is the co-cone of $(\delta^m[-(m-n)], 0, \dots, 0) : I^{[n, m]} \rightarrow J^{m+1}[-(m-n)]$ and that the associated morphism $I^{[n, m+1]} \rightarrow I^{[n, m]}$ coincides with $\varphi_n^{m+1, m}$. Then by Lemma 7.16.(1), we obtain the following commutative diagram

$$\begin{array}{ccccc} M^n & \xrightarrow{\pi_n^m} & I^{[n, m]} & \xrightarrow{\xi_n^m} & \text{coker}(\pi_n^m) \\ \downarrow \pi_n^{m+1} & & \parallel & & \downarrow \eta_n^m \\ & & I^{[n, m]} & \xrightarrow{(\rho^m[-(m-n)], 0, \dots, 0)} & M^{m+1}[-(m-n)] \\ & & \parallel & & \downarrow \lambda^{m+1}[-(m-n)] \\ I^{[n, m+1]} & \xrightarrow{\varphi_n^{m+1, m}} & I^{[n, m]} & \xrightarrow{(\delta^m[-(m-n)], 0, \dots, 0)} & J^{m+1}[-(m-n)]. \end{array}$$

Then by Lemma 7.16, we have the following sequence of morphisms

$$\begin{aligned} \text{coker}(\pi_n^{m+1}) &\cong c(\lambda^{m+1}[-(m-n)] \circ \eta_n^m)[-1] \\ &\xrightarrow{\sim} c(\lambda^{m+1}[-(m-n)])[-1] \\ &\xrightarrow{\sim} \text{coker}(\lambda^{m+1})[-(m-n+1)] \cong M^{m+2}[-(m+1-n)] \end{aligned}$$

where for $a = 1, 2, 3$ we obtain the i -th morphisms by Lemma 7.16 (a). Note that to show that the 3-ed morphism is a trivial injective fibration, we use the fact that M^{m+2} is injectively fibrant. We denote by η_n^{m+1} the above composition. Then it follows from Lemma 7.16 that $\eta_n^{m+1} \circ \xi_n^{m+1} = (\rho^{m+1}[-(m+1-n)], 0, \dots, 0)$. \square

Note that for $\ell > m > n$ we have the following commutative diagram

$$(7-19) \quad \begin{array}{ccccc} M^n & \xrightarrow{\pi_n^m} & I^{[n, m]} & \xrightarrow{\xi_n^m} & \text{coker}(\pi_n^m) \\ \downarrow \pi_n^\ell & & \parallel & & \downarrow \eta_n^m \\ & & I^{[n, m]} & \xrightarrow{(\rho^m[-(m-n)], 0, \dots, 0)} & M^{m+1}[-(m-n)] \\ & & \parallel & & \downarrow \pi_{m+1}^\ell[-(m-n)] \\ I^{[n, \ell]} & \xrightarrow{\varphi_n^{\ell, m}} & I^{[n, m]} & \xrightarrow{\quad} & I^{[m+1, \ell]}[-(m-n)] \end{array}$$

where the term $I^{[n, \ell]}$ is obtained as the co-cone of the bottom right arrow $I^{[n, m]} \rightarrow I^{[m+1, \ell]}[-(m-n)]$.

Proof of Lemma 7.14. By Lemma 2.1, it is enough to show that in the derived category $\mathcal{D}(\mathcal{A})$ there exists a morphism $\psi : I^n \rightarrow K$ for some n such that $k = \psi \circ \pi^n$.

First we recall the definition of the n -generated thick subcategory $\langle J \rangle_n^{\mathcal{D}}$ introduced in [30]. Let \mathcal{T} be a triangulated category. For a full subcategory \mathcal{S} of \mathcal{T} we denote by $\langle \mathcal{I} \rangle_1^{\mathcal{D}}$ the smallest full subcategory of \mathcal{T} containing \mathcal{I} which is closed under taking shifts, finite direct sums, direct summands and isomorphisms. For full subcategories \mathcal{S} and \mathcal{R} of \mathcal{T} we denote by $\mathcal{S} * \mathcal{R}$ the full subcategory of \mathcal{T} consisting of those objects $T \in \mathcal{T}$ such that there exists an exact triangle $S \rightarrow T \rightarrow R \rightarrow$ with $S \in \mathcal{S}$ and $R \in \mathcal{R}$. Set $\mathcal{S} \diamond \mathcal{R} := \langle \mathcal{S} * \mathcal{R} \rangle_1^{\mathcal{D}}$. For $n \geq 2$ we define inductively

$$\langle \mathcal{S} \rangle_n^{\mathcal{D}} := \langle \mathcal{S} \rangle_{n-1}^{\mathcal{D}} \diamond \langle \mathcal{S} \rangle_1^{\mathcal{D}}.$$

Note that $\langle \mathcal{S} \rangle^{\mathcal{D}} = \bigcup_{n \geq 1} \langle \mathcal{S} \rangle_n^{\mathcal{D}}$.

Let $k : M \rightarrow K$ be a morphism in $\mathcal{D}(\mathcal{A})$ with $K \in \langle J \rangle^{\mathcal{D}}$. We claim that if $K \in \langle J \rangle_{n+1}$ then there exists morphism $\pi^n \rightarrow k$. We prove this claim by induction on n . In the case when $n = 0$, K is a direct summand of a finite direct sum of J . Hence it is a cohomologically injective object. Since by definition the morphism $\pi^0 : M \rightarrow J^0$ is cohomologically monomorphism, we have a morphism $\pi^0 \rightarrow k$. We assume that the claim is proved for n . We may assume that K fits into an exact triangle $K \rightarrow K^n \rightarrow K^0 \rightarrow K[1]$ for some $K^n \in \langle J \rangle_{n+1}^{\mathcal{D}}$ and $K^0 \in \langle J \rangle_1^{\mathcal{D}}$. By induction hypothesis, the composite morphism $M \xrightarrow{k} K \rightarrow K^n$ factors through π^n . We set $M^{n+1} := \text{im}(d^n)$. Then we have the following commutative diagram except for the dotted arrow:

$$\begin{array}{ccccccc} M & \xrightarrow{\pi^n} & I^n & \longrightarrow & M^{n+1} & \longrightarrow & M[1] \\ \downarrow k & & \downarrow & & \downarrow k' & & \downarrow k[1] \\ K & \longrightarrow & K^n & \longrightarrow & K^0 & \longrightarrow & K[1]. \end{array}$$

where the top and bottom rows are exact. By an axiom of triangulated category, we have a morphism $k' : M^n \rightarrow K^0$ which complete the above commutative diagram. Since the morphism $\lambda^{n+1} : M^{n+1} \rightarrow J^{n+1}$ constructed in the resolution (7-17) is cohomologically monomorphism, there exists a morphism $h' : J^{n+1} \rightarrow K^0$ such that $k' = h' \circ \iota^{n+1}$.

Recall that I^{n+1} is the co-cone of the composite morphism $I^n \rightarrow M^{n+1} \xrightarrow{\lambda^{n+1}} J^{n+1}$. We have the following commutative diagram except for the dotted arrow:

$$\begin{array}{ccccc} I^n & \longrightarrow & M^{n+1} & \longrightarrow & M[1] \\ \parallel & & \downarrow \lambda^{n+1} & & \downarrow \pi^{n+1}[1] \\ I^n & \longrightarrow & J^{n+1} & \longrightarrow & I^{n+1}[1] \\ \downarrow & & \downarrow h' & & \downarrow h \\ K^n & \longrightarrow & K^0 & \longrightarrow & K[1] \end{array} \quad \begin{array}{c} \curvearrowright \\ k[1] \end{array}$$

Since $I^n[1]$ is a homotopy fiber co-product of J^{n+1} and $M[1]$ under M^{n+1} , there exists a morphism $h : I^{n+1}[1] \rightarrow K[1]$ which complete the above commutative diagram [26, Section 1.3]. Therefore the morphism $h[-1] : I^{n+1} \rightarrow K$ give a desired factorization. \square

In a similar way, we can prove more general statement by using the diagram (7-19).

Lemma 7.17. *Let $k : M^n \rightarrow K$ be an injectively cofibrant morphism of dg \mathcal{A} -modules with $K \in \langle J \rangle^{\mathcal{C}}$. Then there exists $m \geq n$ such that there exists a morphism $\psi : I^{[n,m]} \rightarrow K$ which satisfies $k = \psi \circ \pi_n^m$.*

To finish the proof of Proposition 7.13, in the rest of this section we devote to prove that the simplicial set $N(\Phi)_{/k}$ is weakly contractible.

We will show that any simplicial map $f : X \rightarrow N(\Phi)_{/k}$ with finite simplicial set X factor $*$ in the homotopy category $\text{Ho}(\text{sSet})$. By the same method of the proof of Lemma 6.8 we may assume that the composition morphism $f_1 = \text{pr}_1 \circ f : X \rightarrow N(\mathcal{I})$ is a constant morphism with the value, say, n , and that the map f factors $\text{Hom}^R_{N(\mathcal{U})}(\pi^n, k)$.

$$\begin{array}{ccccc}
 X & & & & \\
 \searrow f' & \nearrow f & & & \\
 & \text{Hom}^R_{N(\mathcal{U})}(\pi^n, k) & \longrightarrow & N(\Phi)_{/k} & \xrightarrow{\text{pr}_2} N(\mathcal{U})_{/k} \\
 & \downarrow & & \downarrow \text{pr}_1 & \downarrow \\
 & \{n\} & \longrightarrow & N(\mathcal{I}) & \xrightarrow{\Phi} N(\mathcal{U})
 \end{array}$$

Let ℓ be an integer greater than n . We denote by $[\mathbf{uni}^{\ell, n}]$ the simplicial map $\Delta^1 \rightarrow N(\mathcal{I})$ corresponding to the morphism $\mathbf{uni}^{\ell, n}$. Let H be the composite map

$$H : X \times \{1\} \xrightarrow{\alpha \times \iota_1} \Delta^0 \times \Delta^1 \cong \Delta^1 \xrightarrow{[\mathbf{uni}^{\ell, n}]} N(\mathcal{I}).$$

where $\alpha : X \rightarrow \Delta^0$ is a unique map. We can check that the following diagram is commutative except the dotted arrow

$$\begin{array}{ccc}
 X \times \{1\} & \xrightarrow{\sim} X & \xrightarrow{f} N(\Phi)_{/k} \\
 \alpha \times \iota_1 \downarrow & \nearrow H' & \downarrow \text{pr}_1 \\
 X \times \Delta^1 & \xrightarrow{H} & N(\mathcal{I})
 \end{array}$$

By the same consideration as in the proof of Lemma 6.8, we see that there exists a dotted arrow which completes above commutative diagram. Observe that the composite map

$$\text{pr}_1 \circ H' \circ (1_X \times \iota_0) : X \times \{0\} \rightarrow X \times \Delta^1 \rightarrow N(\Phi)_{/k} \rightarrow N(\mathcal{I})$$

factors through $\{\ell\} \rightarrow N(\mathcal{I})$. Since $\text{Hom}^R(\pi^\ell, k)$ is the fiber product $\{\ell\} \times_{N(\mathcal{U})} N(\mathcal{U})_{/k}$, the composite map $H' \circ (1_X \times \iota_0)$ is decomposed into a map $g' : X \rightarrow \text{Hom}^R(\pi^\ell, k)$ followed by a canonical map $\text{Hom}^R(\pi^\ell, k) \rightarrow N(\Phi)_{/k}$. We define $\text{Hom}^R(\varphi^{\ell, n}, k)$ to be the fiber product $\{\mathbf{uni}^{\ell, n}\} \times_{N(\mathcal{U})} N(\mathcal{U})_{/k}$. Then we have the commutative diagram

$$\begin{array}{ccccc}
 X \times \{0\} & \xrightarrow{1_X \times \iota_0} & X \times \Delta^1 & \xleftarrow{1_X \times \iota_1} & Z \times \{1\} \\
 g' \downarrow & & H' \downarrow & & \downarrow f' \\
 \text{Hom}^R(\pi^\ell, k) & \longrightarrow & \text{Hom}^R(\varphi^{\ell, n}, k) & \longleftarrow & \text{Hom}^R(\pi^n, k)
 \end{array}$$

By [20] for $\ell \geq n$ we have the commutative diagram

$$\begin{array}{ccc}
 X & \xrightarrow{f'} & \text{Hom}^R(\pi^n, k) \\
 & \searrow g' & \downarrow (\varphi^{\ell, n})^* \\
 & & \text{Hom}^R(\pi^\ell, k)
 \end{array}$$

where the vertical arrow $(\varphi^{\ell,n})^*$ is the morphism induced from the morphism $\varphi^{\ell,n} : \pi^\ell \rightarrow \pi^n$. Therefore it is enough to prove that for some $\ell \geq n$ we have the following homotopy commutative diagram in \mathbf{sSet}

$$\begin{array}{ccc} X & \xrightarrow{f'} & \mathrm{Hom}^R(\pi^n, k) \\ \downarrow & & \downarrow (\varphi^{\ell,n})^* \\ * & \longrightarrow & \mathrm{Hom}^R(\pi^m, k). \end{array}$$

(For simplicity we denote by $[-, +] = \mathrm{Hom}_{\mathbf{hSet}}(-, +)$ the Hom set of the homotopy category \mathbf{hSet} .) In other words, for every element f' of $[X, \mathrm{Hom}^R(\pi^n, k)]$ there exists a natural number $\ell \geq n$ such that the image of f' by the induced morphism $[X, \mathrm{Hom}^R(\pi^n, k)] \rightarrow [X, \mathrm{Hom}^R(\pi^\ell, k)]$ lies in the image of the morphism $[*, \mathrm{Hom}^R(\pi^\ell, k)] \rightarrow [X, \mathrm{Hom}^R(\pi^\ell, k)]$ induced from a unique map $X \rightarrow *$.

By [20, 2.2] we have a weak homotopy equivalence $\mathrm{Hom}^R(\pi^n, k) \simeq \mathrm{Map}_{\mathcal{U}}(\pi^n, k)$. On the other hand we have a natural isomorphism $\mathrm{Map}_{\mathbf{sSet}}(X, \mathrm{Map}_{\mathcal{U}}(\pi^n, k)) \cong \mathrm{Map}_{\mathcal{U}}(\pi^n, k^X)$. Therefore we have a natural isomorphism $[X, \mathrm{Hom}^R(\pi^n, k)] \cong \mathrm{Hom}_{\mathbf{hU}}(\pi^n, k^X)$. Hence it is enough to show that for every morphism $\alpha : \pi^n \rightarrow k^X$ in \mathcal{U} there exists a natural number $\ell \geq n$ such that we have the following homotopy commutative diagram in \mathcal{U}

$$(7-20) \quad \begin{array}{ccc} \pi^\ell & \longrightarrow & k \\ \varphi^{\ell,n} \downarrow & & \downarrow \beta \\ \pi^n & \xrightarrow{\alpha} & k^X \end{array}$$

where we denote by β the morphism induced from a unique map $X \rightarrow *$.

By Lemma 7.17 there exists a morphism $\beta' : \pi^m \rightarrow k$ in \mathcal{U} . Replacing m with $\max\{m, n\}$, we may assume that $m \geq n$. We set $\alpha'' := \alpha \circ \varphi^{m,n}$ and $\beta'' := \beta \circ \beta'$. We also denote by α'' and β'' the morphisms $I^m \rightarrow K^X$, $I^m \rightarrow K^X$ between the co-domains respectively. We define an object $k_1 : M \rightarrow K_1$ of \mathcal{U} to be a morphism $(0, {}^t(\pi^m, \pi^m)) : M \rightarrow \mathrm{cc}(\alpha'', -\beta'')$. Since $(\alpha'', -\beta'') \circ {}^t(1_{I^m}, 1_{I^m}) \circ \pi^m = \alpha'' \circ \pi^m - \beta'' \circ \pi^m = k^X - k^X = 0$, there exists a morphism $\zeta : \mathrm{coker}(\pi^m) \rightarrow K^X$ which completes the following commutative diagram

$$(7-21) \quad \begin{array}{ccccc} M & \xrightarrow{\pi^m} & I^m & \xrightarrow{\xi^m} & \mathrm{coker}(\pi^m) \\ \downarrow k_1 & & \downarrow {}^t(1_{I^m}, 1_{I^m}) & & \downarrow \zeta \\ K_1 & \xrightarrow{\gamma} & I^m \oplus I^m & \xrightarrow{(\alpha'', -\beta'')} & K^X. \end{array}$$

where γ is a canonical morphism for the co-cone construction 2-3. We set $\gamma_a := \mathrm{pr}_a \circ \gamma : K_1 \rightarrow I^m \oplus I^m \rightarrow I^a$ for $a = 1, 2$. Then we have an equation $\pi^m = \gamma_a \circ k_1$ of morphisms in $\mathcal{C}(\mathcal{A})$. Therefore γ_a induces a morphism $k_1 \rightarrow \pi^m$, which will also be denoted by γ_a . In the same way of the proof of Lemma 2.13, from the above diagram (7-21) we obtain the homotopy commutative diagram and an equation of morphisms in \mathbf{hU}

$$(7-22) \quad \begin{array}{ccc} k_1 & \xrightarrow{\gamma_2} & \pi^m \\ \gamma_1 \downarrow & & \downarrow \beta'' \\ \pi^m & \xrightarrow{\alpha''} & k^X, \end{array} \quad \alpha'' \circ \gamma_1 = \beta'' \circ \gamma_2.$$

We denote by $\pi^m \oplus \pi^m$ the object ${}^t(\pi^m, \pi^m) : M \rightarrow I^m \oplus I^m$ of \mathcal{U} . We define a morphism $\delta : \pi^m \rightarrow \pi^m \oplus \pi^m$ in \mathcal{U} to be the morphism induced from ${}^t(1_{I^m}, 1_{I^m}) : I^m \rightarrow I^m \oplus I^m$. Now we have

equations of morphisms in \mathcal{U} .

$$(7-23) \quad \begin{aligned} 1_{\pi^m} &= \text{pr}_1 \circ \delta, \quad \text{pr}_1 \circ \gamma = \gamma_1, \\ 1_{\pi^m} &= \text{pr}_2 \circ \delta, \quad \text{pr}_2 \circ \gamma = \gamma_2. \end{aligned}$$

By Lemma 7.15 we have a injective trivial fibration $\eta^m : \text{coker}(\pi^m) \xrightarrow{\sim} M^{m+1}[-m]$. Since every object of $\mathcal{C}(\mathcal{A})$ is injectively cofibrant, therefore using the lifting property, we can check that the morphism η^m splits. Hence $\text{coker}(\pi^m)$ is isomorphic to $M^{m+1}[-m] \oplus N$ for some weakly contractible module N . From now we identify $\text{coker}(\pi^m)$ with $M^{m+1} \oplus N$. By Lemma 7.15 the first component ξ_1^m of $\xi^m : I^m \rightarrow \text{coker}(\pi^m) \cong M^{m+1}[-m] \oplus N$ is equal to $(\rho^m, 0, \dots, 0)$. Let $\zeta_1 : M^{m+1}[-m] \rightarrow K^X$ be the first component of $\zeta : \text{coker}(\pi^m) \rightarrow K^X$. Then by Lemma 7.17 there exists a natural number $\ell \geq m+1$ such that there exists a morphism $\psi : I^{[m+1, \ell]}[-m] \rightarrow K^X$ which satisfies $\zeta_1 = \psi \circ \pi_{m+1}^\ell[-m]$. We denote by ω the morphism

$$\begin{pmatrix} \pi_{m+1}^\ell[-m] & 0 \\ 0 & 1_N \end{pmatrix} : M^{m+1}[-m] \oplus N \rightarrow I^{[m+1, \ell]}[-m] \oplus N$$

Then we have the equation $\zeta = (\zeta_1, \zeta_2) = (\psi, \zeta_2) \circ \omega$. We denote by K_2 the co-cone of $\omega \circ \xi^m : I^m \rightarrow I^{[m+1, \ell]}[-m] \oplus N$. We set $K_2 := \text{cc}(\omega \circ \xi^m)$. By Lemma 7.16 the morphism ${}^t(0, \pi^m) : M \rightarrow K_2$ of graded modules become a morphism in $\mathcal{C}(\mathcal{A})$, which will be denoted by k_2 . Now we have the following commutative diagram except dotted arrow.

$$(7-24) \quad \begin{array}{ccccc} M & \xrightarrow{\pi^m} & I^m & \xrightarrow{\xi^m} & M^{m+1}[-m] \oplus N \\ \downarrow k_2 & & \parallel & & \downarrow \omega \\ K_2 & \xrightarrow{\gamma'} & I^m & \xrightarrow{\omega \circ \xi^m} & I^{[m+1, \ell]}[-m] \oplus N \\ \downarrow \psi' & & \downarrow (1_{I^m}, 1_{I^m}) & & \downarrow (\psi, \zeta_2) \\ K_1 & \xrightarrow{\gamma} & I^m \oplus I^m & \xrightarrow{(\alpha'', -\beta'')} & K^X \end{array}$$

(A curved arrow k_1 goes from M to K_1 , and a curved arrow ζ goes from $M^{m+1}[-m] \oplus N$ to K^X .)

Since K_1 and K_2 are defined to be the co-cones, a morphism $\psi' : K_2 \rightarrow K_1$ which completes the above diagram is induced from the lower right square. The above diagram gives the following commutative diagram and an equation of morphisms in \mathcal{U}

$$(7-25) \quad \begin{array}{ccc} k_2 & \xrightarrow{\gamma'} & \pi^m \\ \psi' \downarrow & & \downarrow \delta \\ k_1 & \xrightarrow{\gamma} & \pi^m \oplus \pi^m, \end{array} \quad \delta \circ \gamma' = \gamma \circ \psi'.$$

On the other hand, since $(1, 0) \circ \omega = (\pi^m[-m], 0)$, we have the following commutative diagram except the dotted arrow in $\mathcal{C}(\mathcal{A})$

$$(7-26) \quad \begin{array}{ccccc} M & \xrightarrow{\pi_{m+1}^\ell} & I^m & \xrightarrow{\xi^m} & M^{m+1}[-m] \oplus N \\ \downarrow k_2 & & \parallel & & \downarrow \omega \\ K_2 & \xrightarrow{\gamma'} & I^m & \xrightarrow{\omega \circ \xi^m} & I^{[m+1, \ell]}[-m] \oplus N \\ \downarrow \psi'' & & \parallel & & \downarrow (1, 0) \\ I^\ell & \xrightarrow{\varphi^{\ell, m}} & I^m & \xrightarrow{((\pi_{m+1}^\ell \circ \rho^m)[-m], 0, \dots, 0)} & I^{[m+1, \ell]}[-m] \end{array}$$

(A curved arrow π^ℓ goes from M to I^ℓ , and a curved arrow $(\pi^m[-m], 0)$ goes from $M^{m+1}[-m] \oplus N$ to $I^{[m+1, \ell]}[-m]$.)

Recall that K_2 and I^ℓ are given as the co-cones, a morphism $\psi'' : K_2 \rightarrow I^\ell$ which completes the above diagram is induced from the lower right square. Note that since N is weakly contractible, the induced morphism $\psi'' : K_2 \rightarrow I^\ell$ is quasi-isomorphism. By Lemma 2.2 the morphism $\psi'' : k_2 \rightarrow \pi^\ell$ is weak homotopy equivalence. The above diagram (7-26) gives the following commutative diagram and an equation in \mathcal{U}

$$(7-27) \quad \begin{array}{ccc} k_2 & \xrightarrow{\gamma'} & \pi^m \\ \psi'' \downarrow & & \parallel \\ I^\ell & \xrightarrow{\varphi^{\ell,m}} & \pi^m \end{array} \quad \varphi^{\ell,m} \circ \psi'' = \gamma'.$$

Combining the equations (7-23,7-22,7-25,7-27), we obtain a homotopy commutative diagram in \mathcal{U}

$$(7-28) \quad \begin{array}{ccc} \pi^\ell & \longrightarrow & \pi^m \\ \varphi^{\ell,m} \downarrow & & \downarrow \beta'' \\ \pi^m & \xrightarrow{\alpha''} & k^X. \end{array}$$

We explain the combining process. First note that the morphism ψ'' is a weak equivalence in \mathcal{U} . Therefore from the equation (7-27), we obtain an equation $\varphi^{\ell,m} = \gamma' \circ \psi''^{-1}$ in the homotopy category $\mathbf{h}\mathcal{U}$. From the equations (7-23, 7-25) we deduce the equation in the first line below for $a = 1, 2$. Finally, using the equation (7-22) we obtain the equation in the bottom line.

$$\begin{aligned} \varphi^{\ell,m} &= \text{pr}_a \circ \delta \circ \gamma' \circ \psi''^{-1} = \text{pr}_a \circ \gamma \circ \psi' \circ \psi''^{-1} = \gamma_a \circ \psi' \circ \psi''^{-1}, \\ \alpha'' \circ \varphi^{\ell,m} &= \alpha'' \circ \gamma_1 \circ \psi' \circ \psi''^{-1} = \beta \circ \gamma_2 \circ \psi' \circ \psi''^{-1} = \beta'' \circ \varphi^{\ell,m}. \end{aligned}$$

Recalling the definitions of α'', β'' , we see that the diagram (7-28) gives a desired diagram (7-20). Now the proof of Theorem 7.7 is completed.

A Conceptually this paper is very simple.

This paper is lengthy, because we need to work with homotopy theory. However the ideas behind Main theorem and applications are so simple that we require the reader to have knowledge of elementary category theory and homological algebra.

A.1 Main Theorem

We explain that an elementary observation leads to the main theorem 1.2 of this paper.

We use the notations in Introduction. So we have the duality over J .

$$(-)^\circledast := \mathbb{R} \text{Hom}_{\mathcal{A}}(-, J) : \mathcal{D}(\mathcal{A}) \rightleftarrows \mathcal{D}(\mathcal{E})^{\text{op}} : \mathbb{R} \text{Hom}_{\mathcal{E}}(-, J) =: (-)^\circledast$$

We denote by $\langle J \rangle$ the smallest thick subcategory containing J . We claim that if K belongs to $\langle J \rangle$, then the evaluation map $\varepsilon_K : K \rightarrow K^{\circledast\circledast}$ is an isomorphism. Indeed for the case $K = J$ is clear. Since the bi-dual $(-)^{\circledast\circledast}$ is an exact functor, we can check the claim for general $K \in \langle J \rangle$.

We fix a dg \mathcal{A} -module M . It follows from the above claim that every morphism $k : M \rightarrow K$ with $K \in \langle J \rangle$ factors through $\epsilon_M : M \rightarrow M^{\circledast\circledast}$.

$$\begin{array}{ccc} M & \xrightarrow{\epsilon_M} & M^{\circledast\circledast} \\ \downarrow k & \nearrow k^\vee & \\ K & & \end{array}, \quad \begin{array}{ccc} M & \xrightarrow{\epsilon_M} & M^{\circledast\circledast} \\ \downarrow k & & \downarrow k^{\circledast\circledast} \\ K & \xleftarrow[\cong]{\epsilon_K^{-1}} & K^{\circledast\circledast} \end{array}$$

It seems that the derived bi-dual module $M^{\otimes\otimes}$ satisfies one of the two conditions of the limit of the family $M \rightarrow K$ of morphisms. In the following way, we can catch a glimpse of the other condition that we can reach from $K \in \langle J \rangle$ to $M^{\otimes\otimes}$.

It is well-known that a dg-module is obtained as a filtered homotopy colimit perfect modules. Hence the dg \mathcal{E} -module M^\otimes is quasi-isomorphic to the homotopy colimit of some family $\{P_\lambda\}_{\mathbf{L}}$ of perfect \mathcal{E} -modules.

$$M^\otimes \simeq \operatorname{hocolim}_{\mathbf{L}} P_\lambda$$

Applying the dual functor $(-)^{\otimes}$ to this quasi-isomorphism, we obtain the quasi-isomorphisms

$$M^{\otimes\otimes} \simeq (\operatorname{hocolim}_{\mathbf{L}} P_\lambda)^{\otimes} \simeq \operatorname{holim}_{\mathbf{L}} (P_\lambda^{\otimes}).$$

It is clear that $\mathcal{E}^{\otimes} \simeq J$. Therefore, since P_λ is a perfect \mathcal{E} -module, the dual P_λ^{\otimes} belongs to $\langle J \rangle$. This shows that we can reach from $K \in \langle J \rangle$ to $M^{\otimes\otimes}$.

These observations suggest that $M^{\otimes\otimes}$ is the limit of $k : M \rightarrow K$ with $K \in \langle J \rangle$. Actuarially it become true after some modification.

A.2 Localization Theorem

We explain that the view point that a bi-duality is a completion, naturally leads a proof of Theorem 1.10.

We discuss the case when \mathcal{A} is an ordinary ring, M an \mathcal{A} -module and J be an injective co-generator of $\operatorname{Mod} \mathcal{A}$. Then the module M has an injective resolution by the products of J

$$0 \rightarrow M \rightarrow J^{\prod \kappa_0} \rightarrow J^{\prod \kappa_1} \rightarrow J^{\prod \kappa_2} \rightarrow \dots$$

We can reduce the problem the following theorem by setting $\kappa := \sup\{\kappa_i \mid i \in \mathbb{Z}\}$.

Theorem A.1. *We take an injective resolution $M \xrightarrow{\sim} J^\bullet$ of M .*

$$(1-29) \quad 0 \rightarrow M \rightarrow J^0 \rightarrow J^1 \rightarrow J^2 \rightarrow \dots$$

Assume that J^i is a direct summand of J . Then the evaluation map $\epsilon_M : M \rightarrow M^{\otimes\otimes}$ is a quasi-isomorphism.

We explain an outline of a proof.

Assumption A.2. *We assume that $\operatorname{holim} = \operatorname{lim}$.*

We denote by I^n the totalization of the n -th truncated resolution.

$$I^n := \operatorname{tot}[J^0 \rightarrow J^1 \rightarrow \dots \rightarrow J^n].$$

Then by assumption the complex I^n belongs to the thick subcategory $\langle J \rangle$ generated by J . Therefore the canonical morphism $\pi^n : M \rightarrow I^n$ belongs to the under category $\langle J \rangle_{M/}$. Moreover we have a canonical morphism $\varphi^{n+1,n} : I^{n+1} \rightarrow I^n$ for $n \geq 0$ which is compatible with π^n .

$$\begin{array}{ccccccc} & & M & & & & \\ & & \downarrow \pi^n & \searrow \pi^{n-1} & & & \\ \dots & \xrightarrow{\varphi^{n+1,n}} & I^n & \xrightarrow{\varphi^{n,n-1}} & I^{n-1} & \xrightarrow{\varphi^{n-1,n-2}} & \dots \end{array}$$

Note that since the limit $\lim_{n \rightarrow \infty} I^n$ is the totalization of the injective resolution (1-29), the morphisms $\{\pi^n\}$ induces a (quasi-)isomorphism $M \rightarrow \lim_{n \rightarrow \infty} I^n$.

We denote by \mathcal{I} the subcategory of $\langle J \rangle_{M/}$ consisting of objects $\pi^n : M \rightarrow I^n$ and of morphisms $\phi^{m,n} : \pi^m \rightarrow \pi^n$ so that \mathcal{I} is isomorphic to $(\mathbb{Z}_{\geq 0})^{\text{op}}$. Then we have

$$\lim_{\mathcal{I}} \Gamma|_{\mathcal{I}} \cong \lim_{n \rightarrow \infty} I^n \simeq M.$$

Therefore by Theorem 1.2 it is enough to prove that the subcategory $\mathcal{I} \subset \langle J \rangle_{M/}$ is left co-final. Namely for each $k \in \langle J \rangle_{M/}$ the over category \mathcal{I}_k is non-empty and connected.

We recall an elements of Homological algebra: Let M' be another \mathcal{A} -module and $M \xrightarrow{\sim} J^\bullet$ an injective resolution. Assume that an \mathcal{A} -homomorphism $f : M \rightarrow M'$ is given. Then (1) there exists a morphism $\psi : J^\bullet \rightarrow J'^\bullet$ of complexes which completes the commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & J^\bullet \\ f \downarrow & & \downarrow \psi \\ M' & \longrightarrow & J'^\bullet. \end{array}$$

(2) This morphism ψ is not uniquely determined. (3) However it is uniquely determined up to homotopy.

Using the same methods of the proof of (1), we can check that \mathcal{I}_k is non-empty. By the same reason with (2), the category \mathcal{I}_k is *not* connective. However in the same way of the proof of (3) we can verify that \mathcal{I}_k is *homotopy connective*. We explain a little bit more about this in the special case when the co-domain K of $k : M \rightarrow K$ is an injective module:

Since the canonical morphism $\pi^0 : M \rightarrow I^0 = J^0$ is injective, there exists an extension $\psi : I^0 \rightarrow K$ of π^0 . This shows that $\mathcal{I}_k \neq \emptyset$. However there is no canonical choice of an extension. Moreover since the degree 0-part of the canonical morphism $\varphi^{n,0} : I^n \rightarrow I^0$ is the identity map $1_{J^0} : J^0 \rightarrow J^0$, two extensions ψ and ψ' are not connected to each other in \mathcal{I}_k , unless $\psi = \psi'$. Nevertheless we can see that for any two extensions ψ and ψ' there exists a homotopy commutative diagram

$$\begin{array}{ccc} \pi^1 & \xrightarrow{\varphi^{1,0}} & \pi^0 \\ \varphi^{1,0} \downarrow & & \downarrow \psi \\ \pi^0 & \xrightarrow{\psi'} & k \end{array}$$

This shows that the objects ψ and ψ' of \mathcal{I}_k is *homotopically connected* to each other in \mathcal{I}_k . Therefore it is inevitable to work with homotopy theory.

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